All-Pay Contests with Constraints

(Early Draft)

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Abstract

This paper generalizes the results of Siegel (2009) to support contestants who are faced with constraints. It also relaxes the continuity assumptions for some of the players.

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1. Introduction

Often agents make costly irreversible investments to win a prize. In all-pay contests, the players with the highest scores obtain one valuable prize each but the winners' and the losers' costs of effort are at least partially sunk. Siegel (2009) provides closed form formulae for players' expected payoffs for multi-prize complete information all-pay contests under some generic conditions. In this paper we generalize Siegel (2009) to include contests with constraints. Furthermore we relax the restrictions in Siegel (2009) on valuation, cost and score continuity for some of the players.

All-pay contests are used in many areas of research including rent-seeking, political contests, lobbying, patent races, litigation, job tournaments, sports economics, marketing and advertising competition , competition over college seats in selective universities and arms races.[1] In any of these competitive settings, contestants may be faced with constraints. For instance, in the USA a cap on political contributions restricts lobbyists who may be attempting to buy political favors through their political donations. In most of Europe and in Canada politicians and political parties are faced with campaign spending limits. In rent-seeking, in R&D contests and in arms races, participants may have liquidity constraints. In litigation, the plaintiff and the defendant fighting over a favorable court decision have a deadline for collecting evidence. In the labor market, employees aiming to impress for promotion are restricted by a maximum of 24 hours of work in a day. In US professional sports leagues (NBA,NFL, NHL, MLS) teams are constrained with annual salary caps. There are score ceilings in college admissions as one cannot exceed 2400 on the SAT.[2] This paper extends Siegel (2009) to provide generalized expected payoff results in all-pay contests with constraints .

The paper also extends Siegel (2009) to cases where contestants may have valuation, cost and/or score discontinuities . For example, to increase his score the contestant may need to apply for a bank loan which may induce a fixed cost. The Olympic committee may be considering only the number of stadiums promised by potential host cities, so it may not be possible to increase a city's score by less than the cost of a stadium.

Section 2 presents the model. The payoff results in Section 3 are followed by an illustrative application.

2. The Model

Except where otherwise noted we maintain all assumptions in Siegel (2009). In cases where we generalize a named assumption or result in Siegel (2009) we append "generalized" to the name in order to make the changes clear. In cases where we alter the assumption or result but the change is not a strict generalization of the corresponding item in Siegel (2009) we append "modified" to the name. Once the assumptions or results are established and where no confusion results, we drop the epithet.

n players compete for m homogeneous prizes where $0 < m < n$. Each player i simultaneously and independently chooses a score s_i from the set of feasible scores $S_i \subseteq \overline{\mathbb{R}}$. $a_i \epsilon [0, \infty)$ is the initial score of contestant i before he puts forth any effort to improve his score, $a_i = S_i^{\inf} \in S_i$. The initial score gives the degree of the headstart advantage of the contestant.

Each of the m players with the highest scores wins one prize. In the case of ties any tie-breaking rule can be used to allocate the prizes among the tied players. Given a profile of scores $s = (s_1, \cdots, s_n)$, player i's payoff is:

$$u_i(s) = P_i(s)\, v_i(s_i) - (1 - P_i(s))\, c_i(s_i) \tag{1}$$

where $P_i(s)$ is player i's probability of winning at profile s. His payoff if he wins, is given by $v_i(s_i)$. His payoff if he loses is $-c_i(s_i)$. v_i and c_i are defined $\forall\, s_i \in S_i$.

Denote $k_i = S_i^{\sup} \in S_i$ so that for cases where $k_i < \infty$, k_i is a constraint on the player's choice of score.[3] This is the first departure from Siegel (2009) since in Siegel players' sets of feasible scores are not constrained from above. The introduction of a constraint is without loss of generality as the affinely extended real numbers permit the notation $k_i = \infty$ to represent the absence of a constraint. The second departure is that we allow for the possibility of discontinuities in the valuation and/or feasible scores for some of the players. These generalizations allow us to work with players whose actions may be restricted. A player will be said to be *restricted at x* if one of two conditions are satisfied:

a) $x \in S_i$, $v_i(x) > 0$ and either $S_i \cap (x, \infty) = \varnothing$ or $v_i(\min\{s_i \epsilon S_i \mid s_i > x\}) < 0$

b) $x - \varepsilon \in S_i$, $\lim_{z \to x^-} v_i(z) > 0$ and either $S_i \cap [x, \infty) = \varnothing$ or $v_i(\min\{s_i \epsilon S_i \mid s_i \geq x\}) < 0$

A player is restricted at x if he has a positive value from winning at score at x or approaching x from below, but he is either unable to exceed that score or at his next highest feasible score he would have a negative payoff due to discontinuities in his valuation or feasible scores.

These extensions allow payoff characterizations for a rich set of environments which were not covered in Siegel (2009). For instance one can study contests where players face constraints on their actions: such as a trial date in litigation; a common ceiling on political donations; salary caps for sports teams. Note that a common constraint on actions does not necessarily imply that there is a common constraint on scores. However the framework is flexible enough to analyze these competitive environments even when the players face non-identical constraints k_i due to headstart advantages such as policy makers' preferences in rent-seeking, differing degrees of voter loyalty enjoyed by political parties in electoral contests, or differing burdens of proof in litigation. Figures 1 through 3 illustrate the possibilities. For clarity only $v_i(s_i)$ is shown but similar issues with $c_i(s_i)$ are permitted. Figure 1 shows a player with a constraint, the main issue examined in this paper. One can also consider contests where the players have valuation discontinuity. For example, to increase his score the player may need a loan and the application process may entail a fixed cost, see Figure 2. The framework is also equipped to deal with score discontinuities. We require that $a_i \epsilon S_i$ but higher scores may or may not be in S_i. $v_i(s_i)$ and $c_i(s_i)$ need not be defined for infeasible scores, see Figure 3.

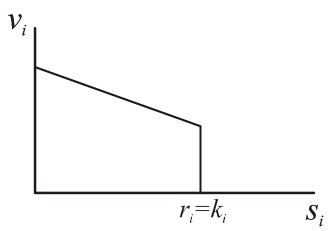

Figure 1

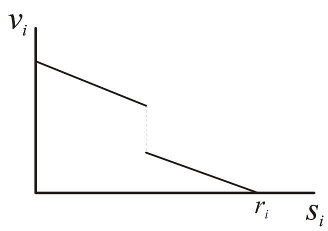

Figure 2

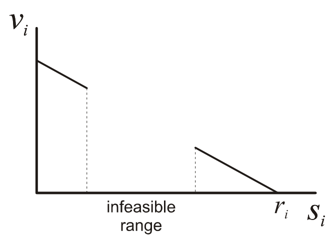

Figure 3

Constraints, as in Figure 1 are permitted for any, none or all players and at any scores. Discontinuities as in Figures 2 and 3 will be permitted in any range of scores only for some of the players. In these environments the paper provides payoff characterizations under the assumptions stated below. First we need to generalize the four main concepts from Siegel (2009) which continue to be key to the analysis.

Definitions:

(i) Player i's *generalized reach*, r_i, is the supremum of the feasible scores at which the player's valuation for winning is non-negative, $r_i = \sup\{s_i \epsilon S_i \mid v_i(s_i) \geq 0\}$. Re-index players in any decreasing order of their reach, so that $r_1 \geq \cdots \geq r_m \geq \cdots \geq r_n$.

(ii) Player $m+1$ is the *marginal player*.

(iii) The *threshold*, T, of the contest is the reach of the marginal player: $T = r_{m+1}$.

(iv) Player i's *generalized power*, w_i, is his valuation for winning at his highest feasible score that is less than or equal to the threshold (if such a score exists in S_i):

$$w_i = \begin{cases} v_i(\max\{s_i \epsilon S_i \mid s_i \leq T\}) & \text{if } a_i \leq T \\ v(a_i) & \text{if } a_i > T \end{cases}$$

The definitions of *reach* and *power* are altered to permit the possibility that a player may be restricted. Note that unlike in the unrestricted case there is no guarantee that the power of players with reaches less than or equal to the threshold will be non-positive. If a player $i > m$ is restricted at his reach, then $w_i > 0$.[4] For instance, consider the contest in Figure 4 with one prize and two players. Player 1 has no constraint while player 2 has a high value for the prize but is financially constrained and cannot achieve a score greater than k_2. The reach of Player 1 is r_1. The reach of Player 2 is k_2. Since $r_1 > k_2$, Player 2 is the marginal player. The threshold of the contest is $T = k_2$. Notice that in the contest in Figure 4 the marginal player has a higher power than player 1, $w_2 > w_1 > 0$, something that cannot happen in the model without constraints or discontinuities.

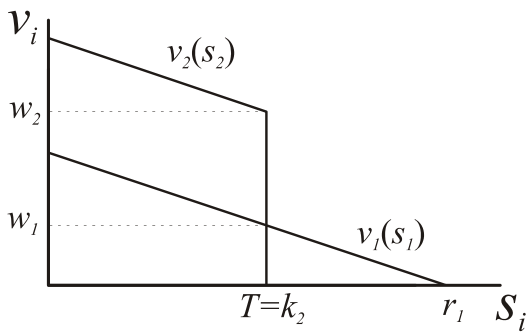

Figure 4

With a slight abuse of nomenclature, define s*core continuity on* $[b,\ d]$ to mean that $[b, d] \cap S_i^c$ has Lebesgue measure zero, *i.e.* scores almost everywhere on $[b, d]$ are in S_i. To reduce unnecessary notation, throughout the paper define the interval [b,d] as the empty set whenever b>d.

Let $N_w = \{1, \cdots, m\}$ denote the set of players with the m (weakly) highest reaches. $N_L = \{m+1, \cdots, n\}$ denotes the set of remaining players whose reaches are less than or equal to the threshold.

Assumptions:

Generalized A1: v_i *and* $-c_i$ *are nonincreasing on* $s_i \epsilon S_i$. *Players in* N_w *have score continuity and continuity of* v_i *and* c_i *on* $[max(a_i, inf\ \{S_{m+1} \cup S_{m+2} \cup \cdots \cup S_n\}), T + \varepsilon]$. *If the marginal player is not restricted at* T, *then he has score continuity on* $[T - \varepsilon,\ T]$.

A2: $c_i(a_i) = 0$, $v_i(a_i) > 0$ *and if* $k_i = \infty$, $\lim_{s_i \to \infty} v_i(s_i) < c_i(a_i)$.

A3: $c_i(s_i) > 0$ *if* $v_i(s_i) = 0$.

Assumptions A2 and A3 are identical to Siegel's. Assumption A1 is weaker than the corresponding assumption in Siegel because it relaxes the score, value and cost continuity requirements for players in N_L (as well as for some ranges of scores for the players in N_w). While the players in N_L have lower reaches than the players in N_w, they are potentially active in the contest and they may have a strictly positive probability of victory. Their valuation and cost structures may influence the equilibrium strategies of all other players including the players in N_w, see Example 1 in Siegel (2009) for an illustration.

The payoff results are valid for games that satisfy the following generalized generic conditions:

(i) *Generalized Power Condition :* The marginal player is the only player with reach at the threshold.

(ii) *Generalized Cost Condition :* If the marginal player is not restricted at the threshold then if $T \in S_{m+1}$ then for every $b \in S_{m+1} \cap [a_{m+1}, T)$, $v_{m+1}(b) > v_{m+1}(T)$, and if $T \notin S_{m+1}$ then for every $b \epsilon \mathrm{S}_{m+1} \cap [a_{m+1}, T)$, $v_{m+1}(b) > \lim_{z \to T} v_{m+1}(z)$.

The Generalized Power Condition parallels Siegel's requirement that the marginal player is the only player with power of 0. However with constraints or discontinuities the marginal player may be restricted at the threshold so there may be no player with zero power. Therefore with constraints the conditions are not equivalent.

Contests that do not meet the generic conditions can be perturbed slightly to meet them. For instance, if there are two players with the same reach at the threshold, giving one of the players a non-negative headstart advantage or the slightest cost advantage can create a contest that meets the power condition. Likewise, perturbing the marginal player's valuation for winning around the threshold leads to a contest that meets the cost condition.

3. Payoff Characterization

In this section we develop the characterization for the expected payoffs in any generic game. This is followed by an example from the literature to show the usefulness of the result. In the following subsection the implications of the payoff characterization are discussed. Then two negative results for non-generic contests are presented.

Three lemmas are used in the payoff characterization.

Modified Least Lemma: *In any equilibrium of a generic contest, the expected payoff of players in* $\mathrm{N_L}$ *is at least zero and the expected payoff of players in* $\mathrm{N_W}$ *is at least their power.*

Proof: In equilibrium no player would choose a score higher than his reach since such a score is either infeasible or would result in negative payoff. By the definition of a player's power and the threshold at most m players can have reach greater than T. Players $i > m$ can guarantee a zero payoff by simply choosing a_i.

Since players $i \leq m$ who have $a_i \leq T$ are not restricted at T and are able to exceed the threshold by ε (Assumption A1), they can at least guarantee an expected payoff equal to their power. Players $i \leq m$ who have $a_i > T$ will win with certainty with $s_i = a_i$ by the Power Condition and hence can guarantee a payoff equal to their power. *Q.E.D.*

To establish the expected payoff of the players Siegel (2009) first establishes the Tie Lemma which shows that if two or more players choose the score b with strictly positive probability, those players either all win with certainty or they all loose with certainty. The Tie Lemma relies on the fact that without constraints or discontinuities if a player's rival has an atom at b and the player can win at b with a positive probability but not with certainty, that player would increase his score slightly to avoid the chance of a tie. However if the player is restricted at b this is not possible.[5] Therefore we proceed by an alternative but related method of establishing the equilibrium payoffs that does not require the Tie Lemma. One useful byproduct of the necessity of bypassing the Tie Lemma is that its proof is what required the strong continuity requirements.

Define for each player G_i as a cumulative probability distribution that assigns probability one to his set of feasible pure strategies S_i. For a strategy profile $G = (G_1, \cdots, G_n)$, $P_i(s_i)$ is player i's probability of winning when he chooses $s_i \in S_i$ and all other players play according to G. Similarly define expected utility $u_i(s_i)$.

Modified Zero Lemma: *In any equilibrium of a generic contest, all players in N_L must have best responses with which they win with probability zero or arbitrarily close to zero. These players have expected payoff of zero.*

Proof: Denote by J a set of players including the m players in N_w plus any one other player $j \in N_L$. Let $\tilde{S}$ be the union of the best-response sets of the players in J and let $s_{\inf}$ be the infimum of $\tilde{S}$. Consider three cases: (i) two or more players in J have an atom at $s_{\inf}$, (ii) exactly one player in J has an atom at $s_{\inf}$, and (iii) no players in J have an atom at $s_{\inf}$.

Case i. Initially denote $N' \subseteq J$ as the set of all players in J with an atom at $s_{\inf}$ where $|N'| > 1$. Every player in $J \setminus N'$ chooses scores greater than $s_{\inf}$ with probability 1. Therefore even if every player that is not in J chooses scores strictly below $s_{\inf}$ with probability 1 that leaves one too few prizes to be divided between $|N'|$ players, so winning at s_{inf} with certainty $\forall\, i \in N'$ is not possible. If there are any players in N' with $P_i(s_{\inf}) = 1$ remove them from N' so that $P_i(s_{\inf}) < 1$ $\forall$ $i \epsilon N'$. If $|N'|=1$ then that player i loses with certainty with score $s_{\inf}$ and i's expected payoff cannot be positive. From the Modified Least Lemma and the Generalized Power Condition this player cannot be in N_w so it must be player j. If $|N'| > 1$, then let H be the set $N' \cap N_w$. Since there is only one player in $J \setminus N_w$, $|H| \in \{|N'|-1, |N'|\}$. $P_i(s_{\inf})=0$ is not possible for any $i \epsilon H$ since i would have $u_i(s_{\inf}) \leq 0$ and he must have a positive payoff by the Modified Least Lemma and the Generalized Power Condition. Likewise if player i loses ties with other players in N' with positive probability $P_i(s_{\inf}) \epsilon (0,1)$ is not possible for any $i \in H$ since i can do better by increasing his score slightly above $s_{\inf}$ to avoid ties. Hence every player in H must win every tie with other players in N' at $s_{\inf}$. This is not possible if $|H| = |N'|$ since there are not enough prizes for all the players in N'. Hence $|H| = |N'| - 1$ so $j \epsilon N'$ and j loses all ties with members of N' at $s_{\inf}$. Therefore $P_j(s_{\inf}) = 0$. Since $j \epsilon N'$ and $j \epsilon N_L$ and $P_j(s_{\inf}) = 0$, so $u_i(s_{\inf}) \leq 0$. By the Modified Least Lemma his expected payoff must be zero.

Cases ii and iii. The corresponding proofs in Siegel (2009) apply without modification and establish that in both cases one player $i \in J$ has a best response in which he wins with probability 0 or arbitrarily close to 0 and has a payoff of at most 0. By the Modified Least Lemma i must have a payoff of 0, and by the Generalized Power Condition $i \in N_L$ and so $i = j$.

The above applies for each player $j \in N_L$. *Q.E.D.*

Generalized Threshold Lemma: *In any equilibrium of a generic contest, the players in N_w have best responses that approach or exceed the threshold and, therefore, the players in N_w have an expected payoff of at most their power.*

Proof: The reaches and therefore the supremum of the best responses of players in $N_L \setminus \{m+1\}$ are strictly below the threshold. Therefore there is some $s' < T$ such that $G_i(x) = 1$ for every player i in $N_L \setminus \{m+1\}$ and every $x > s'$. This implies that every player i in N_w has a strictly positive probability choosing scores that meet or exceed the threshold, that is $G_i(x) < 1$ for every $x < T$. Otherwise for some s in (s', T), $G_i(s) = 1$ for all but at most $m-1$ players in $N \setminus \{m+1\}$. But then the marginal player could win with certainty and by the Cost Condition receive a strictly positive payoff by choosing a score of T if he is restricted at T and $T \in S_{m+1}$ or by choosing a score of $T - \varepsilon$ otherwise : A contradiction of the Zero Lemma.

Take a player i in N_w. If $a_i \geq T$ the Lemma is trivially satisfied for player i. Otherwise, because $G_i(x) < 1$ for every $x < T$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of best responses for player i that approach some $z_i \geq T$. Since x_n is a best response for player i, who has a strictly positive payoff by the Least Lemma and the Power Condition, $v_i(x_n) > 0$. So, by Assumptions A1 and A2,

$v_i(x_n) > -c_i(x_n)$. By continuity, $u_i = u_i(x_n) = P_i(x_n)\, v_i(x_n) - (1 - P_i(x_n))\, c_i(x_n) \le v_i(x_n) \underset{x_n \to z_i}{\to} v_i(z_i) \le v_i(T) = w_i$ so every player in N_w obtains an expected payoff of at most his power. *Q.E.D.*

From these intermediate results we can establish the main result of the paper.

Generalized Theorem 1: *In any equilibrium of a generic contest, the expected payoff of each player in N_w is equal to his power which is greater than zero, and the expected payoff of each player in N_L is zero which is less than his power if he is restricted at his reach.*

Proof: The Modified Least Lemma and the Generalized Threshold Lemma establish that players in N_w have expected payoffs equal to their power which is greater than zero by the Generalized Power Condition. The Modified Zero Lemma establishes that the players in N_L have expected payoffs equal to 0. If a player in N_L is not restricted at his reach his power is less than or equal to zero. If he is restricted at his reach his power is greater than zero so his expected payoff is less than his power. *Q.E.D.*

3.1. Discussion of the payoff characterization.

Players' equilibrium strategies will be effected by all players' constraints, conditional investments, discontinuities and the shape of their valuation functions both for winning and losing the contests for all scores below the threshold. However, Theorem 1 shows that players' expected values in any generic contest depend only on the order of their reaches and their value of winning at the threshold, so only players' reaches need to be computed. To illustrate the usefulness of Theorem 1 consider an example from the literature.

Example: Meirowitz (2008) analyzes the sources of incumbency advantage in a first-past-the-post electoral contest with campaign spending where one dollar of campaign spending raises the score of the political candidate by one. The incumbent (I) and the challenger (C) have a common valuation of the prize normalized to 1. The candidates have potentially different marginal utility cost of raising funds, $\beta_i \ \ \forall i \in \{I, C\}$. In the contest without spending limits, Meirowitz (2008) considers a positive headstart advantage $\alpha > 0$ for the incumbent due to existing name recognition. When studying the effect of a spending limit $\overline{m}$ in the presence of such a head-start advantage the analysis only presents the case where the limit is so restrictive that even if the challenger where to spend the maximum permissible amount and the incumbent were to spend zero, the incumbent would win the contest, $\overline{m} < \alpha$. Hence the equilibrium is in pure strategies where no candidate engages in campaign spending.

(i) Application of the Generalized Theorem 1 to spending limits with a headstart advantage, $\overline{m}>\alpha>0$: Extend the Meirowitz (2008) analysis of spending limits using Generalized Theorem 1. An argument against spending limits is that they restrict the challenger's ability to catch up with the incumbent who often enjoys a headstart advantage. Hence limits may benefit the incumbent. However proponents of spending limits suggest that limits will level the playing field in favor of the candidate with lesser resources. Incumbents tend to be more efficient at fund-raising, $\beta_I < \beta_C$. As a sitting officeholder an incumbent is in a position to dispense political favors and hence has better access to resources. Therefore it is often argued that a limit restricts the ability of the incumbent to take full advantage of his fundraising efficiency and hence benefits the challenger.

Generalized Theorem 1 can be applied to show that with any headstart advantage, $\alpha>0$, however small, a spending limit benefits the incumbent no matter how dramatic the difference in fundraising abilities may be. The "headstart advantage" argument of the opponents of spending limits always trumps the "lesser resources" argument of the proponents of limits.

First we need to convert Meirowiz' framework into the notation of this paper. The monetary limit on campaign spending is denoted by $\overline{m}$ and is common to both players. However, since the incumbent has a headstart advantage of α, the constraints on scores are asymmetric: $k_C = \overline{m}$ and $k_I = \alpha + \overline{m}$. The challenger's payoff and cost functions are given by $v_C(s_C) = 1 - \beta_C s_C$ and $c_C(s_C) = \beta_C s_C$ for $s_C \epsilon [0, k_C]$. Since the incumbent starts with a score of α his payoff function is $v_I(s_I) = 1 - \beta_I(s_I - \alpha)$ and $c_I(s_I) = \beta_I(s_I - \alpha)$ for $s_I \epsilon [\alpha, k_I]$. Therefore the reach of the challenger is $r_C = \min\{\overline{m}, 1/\beta_C\}$ and the reach of the incumbent is $r_I = \min\{\alpha + \overline{m}, \alpha + 1/\beta_I\}$.

Without a spending limit, the reach of the incumbent is higher than the reach of the challenger, $1/\beta_C < \alpha + 1/\beta_I$. Hence the challenger is the marginal player. From Generalized Theorem 1 the challenger has zero expected payoff. The threshold is $1/\beta_C$ so the incumbent has an expected payoff of $v_I(1/\beta_C) = 1 - \beta_I(1/\beta_C - \alpha) > 0$.

If the expenditure limit is less than $1/\beta_C$ it becomes binding and $r_C = \overline{m} < \overline{m} + \alpha$. Since r_C is less than the incumbent's reach $r_I = \min\{\alpha + \overline{m}, \alpha + 1/\beta_I\}$ the challenger is still the marginal player and his expected payoff remains zero. However the limit reduces the challenger's reach (the threshold of the game) and hence increases the expected payoff of the incumbent, $v_I(\overline{m}) = 1 - \beta_I(\overline{m} - \alpha) > 0$. Hence the imposition of a spending limit always benefits the incumbent as long as the incumbent has a headstart advantage however small that may be.

Once the expected payoffs of the players are established it is straightforward to derive equilibrium distributions of the players as well as results on expected spending and probability of winning using the standard methods in Hillman and Riley (1989) and Baye *et. al* (1993) or via the algorithm in Pastine and Pastine (2011c).

(ii) Application of Generalized Theorem 1 to spending limits with many candidates and cost discontinuities: Meirowitz (2008) has only two candidates running for election. However in countries where campaign spending limits are in place with a first-pass-the-post system, such as England, often more than two political parties compete.[6] The payoff characterization in Generalized Theorem 1 does not require the full derivation of the equilibrium. Hence we can easily add more candidates and compute which political candidate benefits from a spending limit. Furthermore we can relax some continuity requirements on the value and cost functions. Below we employ the theorem to show that a moderate cap benefits a charismatic but financially challenged third-party candidate, but that a very restrictive cap benefits the incumbent.

Add a third-party candidate to the model described in Application (i) with the same notation. Suppose that the third-party candidate is charismatic and has leadership skills (Candidate L) so that one dollar of campaign spending increases his score by $\eta_L > 1$. We can also incorporate some more realistic fundraising issues. The third-party candidate can raise up to Γ dollars from his core supporters at the marginal utility cost of raising funds $\beta_L = \beta_C > \beta_I$. To raise more than Γ dollars, he can apply for a loan, but the application process imposes a utility cost of *F*. For simplicity assume that once he applies for the loan his marginal utility cost of raising funds remains the same, β_L.[7] Due to the fixed cost of getting the loan the third-party candidate has a cost and value discontinuity. For scores $s_L \le \eta_L \Gamma$ the cost is $\frac{s_L}{\eta_L}\beta_L$. However for scores $s_L > \eta_L \Gamma$ the cost is $\frac{s_L}{\eta_L}\beta_L + F$. As before α is the headstart advantage of the incumbent and all candidates have a common valuation of the prize normalized to 1.

In order to limit the number of cases and to focus the point, we examine the model where the key parameters jointly satisfy the following restrictions: *(i)* $\Gamma < \frac{1}{\eta_L \beta_L}$ and *(ii)* $\alpha < (\eta_L - 1)\Gamma$ and *(iii)* $F \in \left(\frac{\eta_L - 1}{\eta_L},\ 1 - \beta_L \Gamma\right)$. Restriction *(i)* puts an upper bound on the liquidity of the third-party candidate such the candidate will consider applying for a loan. Restriction *(ii)* puts an upper bound on the incumbent's head-start advantage. The parameter restriction *(iii)* means that the fixed utility cost of applying for the loan is high but not prohibitively high such that if the third-party candidate borrows he can still get a positive payoff if he wins.

Without a spending limit $r_L = \eta_L \frac{1-F}{\beta_L}$. With a spending limit each candidates' monetary spending is constrained at $\overline{m}$. For candidate L this translates into a score constraint of $k_L = \eta_L \overline{m}$. Hence the reach of the third-party candidate is $r_L = \min\left\{\eta_L \overline{m},\ \eta_L \frac{1-F}{\beta_L}\right\}$ while the reach of the incumbent and the reach of the challenger are as before. In the absence of a spending limit and under the parameter restrictions above $r_I > r_C > r_L$. The challenger is the marginal player so the threshold is $1/\beta_C$. By Generalized Theorem 1, the incumbent has an expected payoff of

$v_{\mathrm{I}}\left(\frac{1}{\beta_C}\right) = 1 - \beta_I\left(\frac{1}{\beta_C} - \alpha\right) > 0$. The challenger and the third party candidate receive an expected payoff of zero. The challenger is disadvantaged because the incumbent has a head-start advantage and is a better fundraiser. In addition to these disadvantages the third-party candidate has a very high cost of raising large sums of money due to the fixed cost imposed by applying for the loan which puts him at a disadvantage despite his greater effectiveness of campaign spending.

For a common monetary cap $\overline{m} < \frac{1-F}{\beta_L}$, all candidates are restricted at their score caps and the reaches of the candidates are given by $r_I = \alpha + \overline{m}$ and $r_L = \eta_L \overline{m}$ and $r_C = \overline{m}$. If the cap is moderate $\overline{m} \in \left(\frac{\alpha}{\eta_L - 1}, \min\left(\frac{1-F}{\beta_L}, \eta_L \Gamma - \alpha\right)\right)$, then $r_L > r_I > r_C$.[8] The incumbent becomes the marginal player. The threshold of the contest is $\alpha + \overline{m}$. By Theorem 1, the incumbent and the challenger have expected payoffs of zero and the third-party candidate receives an expected payoff of $1 - \beta_L \frac{\alpha + \overline{m}}{\eta_L} > 0$. With a moderate limit, the effective campaign spending of the third-party candidate overwhelms the head-start advantage of the incumbent. Hence a moderate limit hurts the incumbent compared to no restrictions.

If the cap is very restrictive , $\overline{m} \in \left[0, \frac{\alpha}{\eta_L - 1}\right)$, then the order of candidates' reaches is given by: $r_I > r_L > r_C$. The third party candidate is the marginal player and $\eta_L \overline{m}$ is the threshold. By Generalized Theorem 1, the incumbent has the expected payoff $1 - (\eta_L \overline{m} - \alpha)\beta_I > 0$. The challenger and the third-party candidate have expected payoff zero. The head-start advantage of the incumbent overwhelms the campaign spending effectiveness of the third-party candidate with leadership skills. The cap is too restrictive for the third-party candidate to catch up with the incumbent. Note that the expected payoff of the incumbent in this case is higher than the expected payoff he would have had if there were no campaign spending restrictions.

Because players' expected payoffs from the contest depend only on the order of their reaches and on their valuation of winning at the threshold, the striking insight of Siegel (2009) continues to hold in contests with constraints; The players' costs of losing do not affect payoffs. They may have a significant effect on equilibrium strategies, but not on expected payoffs. Similarly, a change in the constraint of any player other than the marginal player does not affect the payoff of any player as long as the change does not alter the identity of the marginal player.

Corollary to Generalized Theorem 1: *In any equilibrium of a generic contest, consider a small change in a player's constraint such that the identity of the marginal player remains the same. A change in the constraint of any player other than the marginal player does not affect the payoff of any player. A change in the marginal player's constraint does not affect the payoff of any player in N_L (including his own). If the marginal player is restricted at k_{m+1} then relaxing his constraint decrease the expected payoff of each player in N_w.*

Proof: All points follow directly from Generalized Theorem 1, the definition of reach and the definition of a player's power. The derivative of all players' expected payoffs with respect to k_j is zero for all $j \neq m+1$. For all $i \in N_L$ the derivative of player i's expected payoff with respect to k_{m+1} is zero. If $T = k_{m+1}$ a marginal decrease in k_{m+1} directly decreases T. A marginal increase in k_{m+1} increases T if player $m+1$ is restricted at T. If $T \neq k_{m+1}$ then marginal changes in k_{m+1} do not alter T. Finally, if $T = k_{m+1}$, then for all $i \epsilon N_w$ the derivative of player i's expected payoff with respect to k_{m+1} *is* $-\frac{\partial v_i(s_i)}{\partial s_i}\Big|_{s_{i=T}} \leq 0$. If $T \neq k_{m+1}$ then the derivatives of all players' expected payoffs with respect to k_{m+1} are zero. *Q.E.D.*

One implication of the Corollary is that a player's expected payoff is affected by a change in his own constraint only if the change in his constraint switches him between N_w and N_L. Other changes in his constraint will potentially affect equilibrium strategies, but they will not affect the player's own payoffs.

3.2 Non-generic contests.

Siegel (2009) presents two results (Corollaries 2 and 3) which hold for non-generic contests. Unfortunately these two corollaries do not extend to non-generic contests with constraints. This can be demonstrated employing an example from the literature.

Consider the non-generic contest in Che and Gale (1998) where two players {A,B} compete for one prize. $v_i(s_i) = V_i - s_i$ and $c_i(s_i) = s_i$ $\forall i \in \{A,B\}$. The players have different valuations of the prize, $V_A > V_B > 0$ and they face a common constraint $k_A = k_B = k$ so $S_i = [0, k]$. Any ties are resolved by coin toss. Che and Gale (1998) shows that for a sufficiently restrictive constraint, $k < V_B/2$, in any equilibrium both players choose $s_i = k$ with probability 1. This contest is non-generic because for low *k*, players *A* and *B* have the same reach $r_A = r_B = k$. There is more than one player whose reach is at the threshold, violating the Power Condition.

Theorem 1 in Siegel (2009) gives the expected payoffs in any continuous generic game and Corollary 2 shows that every continuous non-generic contest has at least one equilibrium in which the payoffs match those described in Theorem 1. The above example shows that this does not generalize to contests with constraints. In Che and Gale (1998), with $k < V_B/2$ the unique equilibrium expected payoff for player i is $0<(V_i/2) - k<w_i$ $\forall\, i \in \{A, B\}$. These expected payoffs are strictly different from either of the possible expected payoffs in Generalized Theorem 1 (either zero or w_i) and hence are violations of a conjectured extension of Siegel's Corollary 2.

Corollary 3 in Siegel (2009) shows that in any continuous contest without constraints where all players are identical, all players have an expected payoff of zero. While the results in Che and Gale (1998) are for players with different valuations, the same logic carries over to identical players facing a common constraint and non-zero probabilities of winning a tie. If the common constraint is sufficiently restrictive, there is an equilibrium where both players choose scores at the constraint with probability one. This yields positive expected payoffs for both players, a violation of a conjectured extension of Siegel's Corollary 3. With constraints, in equilibrium players can put probability mass points at scores where they do not win or lose with certainty. This implies that Siegel's Tie Lemma does not extend to constrained contests .

4. Conclusion

This paper provides a generalization of Siegel (2009) to include constraints on players' choices. In a broad class of all-pay contests, it derives a simple closed-form solution for expected payoffs. These include contests with many players, many prizes, non-ordered costs functions, conditional investments, constraints on players' actions, as well as discontinuous payoff functions and action spaces for some players.

In some applications players' expected payoffs are the main item of interest. For example, one may be concerned about the effect of a policy on the market participants. Or one may be primarily interested in whether agents in a multi-stage game chose to enter a contest. In these cases the results can be used directly eliminating the need for the full derivation of the equilibrium.

In applications where the full characterization of the equilibrium is of interest, finding the players' expected payoffs is a crucial first step in the derivation of the equilibrium. Siegel (2010) uses the expected payoff results in Siegel (2009) to derive a straightforward algorithm for finding the equilibrium in contests without constraints. Pastine and Pastine (2011c) uses the expected payoff results in this paper with constarints to derive a similar algorithm for the derivation of the equilibrium.

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Notes

[1] Among others see Hillman and Samet (1987), Hillman and Riley (1989), Ellingsen (1991), Baye *et. al* (1993), and Konrad (2002) for contests in rent seeking; Che and Gale (1998 and 2006), Kaplan and Wettstein (2006), Meirowitz (2008), Pastine and Pastine (2010a and 2010c) for political contests; Leininger (1991) for R&D races; Bond (2009) for litigation contests; Clark and Riis (1998) for job tournaments; Pastine and Pastine (2010a) for advertising competition; Fu (2006) and Pastine and Pastine (2010b) for affirmative action in college admissions; and O'Neil (1986) for arms races. See Konrad (2009) for an extensive survey on contests.

[2] For all-pay contests with constraints see Che and Gale (1998), Meirowitz (2008), Pai and Vohra (2009), Megidish and Sela (2010), Pastine and Pastine (2010a and 2011b), and Szech (2011) and Leininger (1991) in a complete-information framework. See Che and Gale (1996), Laffont and Robert (1996), Gavious, Moldovanu and Sela (2002), Sahuguet (2006), Kirkegaard (2008), and Pai and Vohra (2009) in a framework with incomplete information. See Rapoport and Amaldoss (2000) for an experimental analysis of all-pay auctions with bid caps and the comment in Dechenaux *et. al* (2006).

[3] Conceptually there are two possible types of constraints: constraints on effort and constraints on scores. Both types of constraint can be captured by this specification. Examples of constraints on effort include liquidity constraints or the maximum permissible donation to a candidate's political campaign in the U.S. Since effort translates directly into scores and the most restrictive possible constraint is zero effort, $k_i \geq a_i$ captures all possibilities. Constraints placed directly on permissible scores are also possible. For example, by construction the maximum possible score that can be achieved on the SAT university entrance exam is 2400. With constraints directly on scores, an initial score higher than the maximum possible score is nonsensical so $k_i \geq a_i$ can be assumed without loss of generality.

[4] It is possible to maintain the definition of power used in the model without constraints: A player's valuation from winning at the threshold. However this would require that $v_i(T)$ and $c_i(T)$ be defined for cases where $T \notin S_i$ which would raise non-trivial conceptual issues. What is the cost of a loan that nobody is willing to extend? Moreover it would not alter the fact that a low cost competitor who is constrained at a score below the threshold would have positive power.

[5] If we revert to Siegel's strong continuity assumptions we can get this Generalized Tie Lemma: "*If two or more players who are not restricted at x have an atom at x then all the players with the atom that are not restricted at x either win with certainty or lose with certainty at x.*" While this is potentially useful, it is not sufficient to proceed as players may well be restricted.

[6] And in at least one case, Ireland, more than one prize is possible. In many Irish political districts the two candidates with the highest vote totals each take a seat in parliament.

[7] In reality the utility cost of paying back the loan is likely to be higher if the candidate loses than if he wins since office holders have more fundraising opportunities than private citizens. An important implication of Theorem 1 is that, although the costs of paying back the loan if he loses will have a significant effect on the equilibrium of the game, they will not effect the expected payoffs. Hence we do not have to specify the utility cost of the debt if the candidate loses, but only need to assume that it is discontinuous only at Γ.

[8] The specified range for $\overline{m}$ is non-empty by parameter restrictions (ii) and (iii). The condition $\overline{m} < \eta_L \Gamma - \alpha$ comes from the Assumption A1 requirement that the players in N_w have score, value and cost continuity on [0, T+ε).