WEAK & STRONG FINANCIAL FRAGILITY

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ABSTRACT. The linkages in terms of returns and volatility between financial service institutions asset returns during periods of crisis is one of two types, depending on whether the returns are either asymptotically independent (weak fragility) or asymptotically dependent (strong fragility), regardless their correlation. If asymptotically independent, the dependency when present, eventually dies out completely at the more extreme quantiles. We study the joint loss behavior of correlated bank portfolios, due to e.g. loan syndication, under the weak assumption that the asset and liability return distribution are in the domain of attraction of an extreme value distribution. Thus we cover both discrete and continuous compounding. It is shown that due to the bank portfolio induced linearity, the type of extreme value distribution to which the marginal distributions are attracted determines whether the fragility is weak or strong and we provide an index for the dependency. This permits a characterization of systemic risk inherent to different bank network structures. The theory also suggests the functional form of the economically relevant copulas.

1. INTRODUCTION

The banking system is inherently fragile and financial crises are a recurrent phenomenon. Economic theory has developed two coherent, not necessarily mutually exclusive, views of how financial crisis arise and spread. At the root of both views is the fact that the intermediation by banks entails lending long while borrowing short through immediately callable deposits. The micro view of financial crises holds that a financial crisis starts at an individual bank and then spreads through the system by various contagion mechanisms. The macro view describes a crisis as originating from the common exposures of banks to diverse macro risk factors, either on the asset or on the liability side

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of banks balance sheets. Empirically, the latter explanation appears to be the more important explanation.[1]

Banking crises are typically regarded as more serious for the economy than crises in other parts of the financial service sector (insurance). Banks provide an important positive externality to the macro economy through their maintenance of the payment and settlement system and by channeling the monetary policy decisions. The clearance system is as strong as its weakest link. The fallout from a bank failure can sort large negative effects. For example the US S&L crisis is estimated to have cost the US tax payer an amount in the order of 4% of GDP, and the Swedish banking crisis of the 1990's cost as much as 10% of GDP. This evidence not withstanding, central banks, ministries of finance, supervisors and regulators are hard pressed when asked to characterize the fragility of their financial system (let alone being able to evaluate the welfare benefits and costs of regulation and crises). Thus, as is forcefully argued by Borio (2003), there is great need for a measure which reflects the amount of systemic risk inherent to a particular banking system.

Empirical research has tried to measure the interdependence within and between financial systems by correlating bank stock returns. The correlation literature is aptly summarized by the informal market speak that during times of crisis there is breakdown of correlation, in the sense that it appears as if all markets suddenly exhibit the same behavior and move down in tandem. This is also picked up by the conditional volatility regression models, often based on a multivariate GARCH structure, which finds that during times of crises the volatility in markets increases. There is some discussion over whether the perception of correlation breakdown is an artefact due to the act of conditioning. On a more general level, Embrechts, McNeil and Strauman (2002) discuss the pitfalls of normal based correlation analysis as a means to measure systemic risk; see the Appendix for some graphical evidence.

Alternatives for the correlation analysis as measures of systemic risk are copulas and multivariate extreme value analysis. Copulas give the dependency structure embedded in the joint distribution function parametrically, see Longin and Solnik (2001) for an early empirical application based on the logistic copula. Extreme value based analysis is a semi-parametric approach which captures the dependency in the tail regions of the joint distribution function . This literature finds evidence for some financial markets to co-depend on each other even in

[1]For a recent literature review of systemic risk issues, see e.g. De Bandt and Hartmann (2001).

the tail area, while other markets appear to become independent at the more extreme levels, see Hartmann et al. (2004) and Poon et al. (2004). In the first case we speak about asymptotic dependence, while in the other case we say that markets are asymptotically independent (which does not require markets to be independent). For the systemic stability, the asymptotic dependence is the issue to worry about. In this paper we ask how this asymptotic dependence may arise under alternative assumptions regarding the marginal distributions of the risk drivers.

If one asks a bank supervisor or bank risk manager to indicate the amount of risk the bank is exposed to, one nowadays readily receives an answer in terms of the Value-at-Risk (VaR) given a certain low loss probability. One can level quite a bit of critique at this measure, but an advantage of this concept is that it has provided a simple and widely understood concept by which risk exposures of different financial institutions can be readily compared. On the systems level, though, we still lack such a simple measure. Somehow one would like to have a measure comparable to the 'Richter scale' for earthquakes which would be indicative of the systemic risk in a banking system. One aim of this paper is to advance a measure of systemic risk, coined the Fragility Index (FI), which has several of the desirable properties in common with the univariate VaR concept. This may enable regulators to make comparisons across countries or different parts of the financial service sector and to determine, at least relatively, their regulatory stance. The measure is close in spirit to the VaR measure, except that it is stated inversely in terms of large loss probabilities, rather than loss levels (this permits for different loss levels for different banks, while still providing a single index number for the entire system). It reflects different possible intensity levels of systems dependence, ranging from perfect dependence, to asymptotic dependence, asymptotic independence and just independence. The measure is not tied to a specific distribution, such as the correlation concept which is so intimately related to the normal distribution. The FI index therefore provides a certain robustness. The paper develops the theoretical concepts and background, but does not enter into detailed statistical measurement, since we can easily refer to the growing body of papers that have used methodologies consistent with our theoretical views.

Our approach exploits the linearity of portfolios driven by the incentives for diversification, as in e.g. loan syndication. This applies both to the asset side as well as to the liability side of insurance or bank balance sheets. We show that popular theoretical economic explanations for systemic breakdowns, such as a macro shock, a sunspot or micro based contagion, fit our portfolio setup. The approach is of semi-reduced form, since it takes as given a certain distribution of the assets and liabilities across the system in characterizing the system's stability. It does not analyze how such a distribution is the outcome of incentives and institutions. The latter question is clearly also of interest, but is outside the scope of a single paper and is well treated elsewhere. The linearity of a bank portfolio in the individual assets or risk drivers, in combination with the univariate properties of asset return distributions provides a classification of financial systems which are strongly fragile and systems which are only weakly fragile. This setup, it is hoped, bridges the gap that exists between the theoretical developments, empirical material and statistical methods.

We consider both continuous and discrete compounding. Under discrete compounding the loss returns have a natural lower limit of minus one. It follows that the limit law for the minimum is in the domain of attraction of the Weibull extreme value distribution. Under continuous compounding, the log-returns have no natural bound and hence are either in the domain of attraction of an extreme value Gumbel or a Frechet distribution. If the tail of the loss distribution is exponential in nature, then the Gumbel limit law applies, whereas a power type decline implies the Frechet limit law. We prove that different portfolios are asymptotically dependent if the individual return distributions are either in the domain of the Weibull or the Frechet limit law, but can be asymptotically independent if the loss distributions are in the domain of attraction of the Gumbel law. For example, the normal distribution is in the domain of attraction of the Gumbel law, so that portfolios are asymptotically independent even though the portfolios induce non-trivial multivariate normal distributions. Therefore such systems only exhibit weak fragility due to the correlation between the portfolios (see the figures in the Appendix). Per contrast uniform (discrete compounding) or Student-t (log-returns) distributed returns, which are respectively in the domain of attraction of the Weibull and the Frechet laws, induce strong fragility. Under multivariate normality, it appears that supervisors can largely confine themselves to independent supervision of financial institutions, whereas if the uniform or Student-t distributions are the empirically relevant distributions the systemic risk is a point of concern which cannot be ignored. To obtain these results we exploit probabilistic results for distributions which are in the domain of attraction of the Frechet limit law. This also is the case for some of the elements in the domain of attraction of the Gumbel distribution. Novel probabilistic results are obtained for the Weibull class before we can get to the fragility results.

Subsequently, the FI measure is used to classify the systemic stability of different banking network structures under alternative probabilistic assumptions. It is shown that rankings based on the correlation structure may give rankings that differ from the measure we prefer. The techniques can also be used to delineate a class of economically relevant copulas. The parametric approach towards joint tail risk by estimating copulas in e.g. portfolio Value-at-Risk exercises is rapidly gaining in popularity. The variety of copulas is 'large' and the question is which types of copulas are relevant for the problem at hand, see the recent contribution by Dunbar in Risk Magazine (October , 2003) who asks: 'Looking for the right copula'. As it happens, copulas are often chosen for their convenience in estimation, but economic criteria have as of yet scarcely received attention when choosing a copula. The ideas we develop in this paper, coupled with existing empirical evidence on the marginal distributions of returns, suggest which types of copulas are relevant for financial and risk management problems.

The structure of the rest of the paper is as follows: In section 2 we discuss discounting and the linearity of bank portfolios and affine fundamentals based models such as the CAPM. A discussion and comparison of different measures to characterize linkages during periods of market stress is provided in section 3. The analytic claims of the paper on the relationship between the risk drivers' marginal tail properties and the degree of tail dependence are in section 4. The cases of weak and strong fragility are treated in separate subsections. Financial economic analysis is given in section 5. Finally, section 6 provides a summary and conclusions.

2. Affine Portfolios and Compounding

Financial institutions are linked in a number of ways. An important linkage is through their mutual exposures, similar investments and liabilities. Take e.g. a reinsurance firm which reinsures part of an insurance policy written by an insurance firm, retrocedes part of this reinsurance contract to another reinsurer and invests the premia it receives on the reinsurance policy in a diversified portfolio of stocks and bonds. All this activity is undertaken to diversify the risk of holding an overly specialized portfolio. The diversification activity on the liability side produces direct linkages within the insurance and reinsurance sector. But since these companies invest the premia in well diversified portfolios, there is also an indirect linkage by the risk factors which drive the market risks. Commercial banks are typically heavily exposed to each other through the interbank market by which the banks manage their liquidity. Typically commercial banks loan to the same sectors in the economy, which again produces the exposure to the same macro risk drivers (through the movements in the value of the received collateral, e.g. house prices in case of mortgages). Syndicated loans whereby several banks underwrite a large loan directly, expose different banks to the same risk. Investment banks often hold large trading portfolios and hold stakes in commercial companies which belong to the clientele of the bank. Last but not least, banks in many countries do also hold sizable cross-participations in each other. In summary, financial service companies hold portfolios which are linked and which are directly or indirectly exposed to the same risks or risk factors.

The portfolio structure of the financial service sector endows it with a linear linkage structure. Insofar banks hold the same assets or finance the same loans, the linearity is direct. For example, if a bank underwrites a syndicated loan, it allocates a percentage of its capital to this loan. Different banks may have a different exposure to the same loan. But all the participating banks in the syndicate have their return on capital linearly related to the return on the syndicated loan. Similarly, when different investment banks hold stakes in the same companies, bank stocks become linearly related. Indirectly, banks' return on capital is also linearly related through their exposure to macro risk factors. For example, while mortgages or small business loans may be uniquely held by a single bank, i.e. are not securitized, most banks do engage in the same activity by lending to similar type of households and businesses. Therefore, insofar collateral value is driven by macro risk factors, all banks are exposed to the same risks. Much of the finance theory, assumes that returns are linearly related to the macro risk factors. The CAPM is the prime example, but other theories like the APT also are linear in their factors. Below we give some other examples. The linear structure of our model makes it comparable with the setup of theoretical crises models like in Allen and Gale (2000), Freixas, Parigi and Rochet (2000). We elaborate on this further in the last section.

This is not to say, there are no cases where the relationship between the returns and factors is non-linear. For example, Morris and Shin (2001) present a theoretical model of bank runs with noisy signals that eliminate the multiplicity of sunspot equilibria, but where the returns are a non-linear function of the fundamentals. Another example is a portfolio which both contains options and the underlying stocks. Option theory holds that both are linearly related to the market factor, where in case of the option also the elasticity of the option with respect to the market factor enters. But this is only a local result and the relationship between the returns on the stock and option are non-linear due to the fact that the option may be out of the money at expiration. In the last section we discuss an example of such a case. For the greater part we will proceed on the assumption that the interdependence between the financial institutions' returns is linear. To show the scope of our linear modelling strategy, we turn to one other example which is explicitly macro in its orientation.

2.0.1. *affine forex models.* Finacial crisis are not the limited to the banking sector. The past decade has seen several financial crises on an international scale, whereby different countries' financial crises were linked through their exchange rates. The simple monetary model of the exchange rate provides such a linkage through its log-linearity in the macro fundamentals. The standard monetary model of the log price of currency j in terms of currency 0 is derived form the log differences of the country's quantity equations

$$\begin{aligned} s_{0j} &= (m_0 - \phi y_0 + \lambda R_0) - (m_j - \phi y_j + \lambda R_j) \\ &= g_0 - g_j, \qquad j = 1, \cdots, n, \end{aligned}$$

say, where g_0 and g_j are composite fundamentals consisting of the logarithmic money stock measure m, the negative of the income elasticity times log real income $-\phi y$ and the semi interest rate elasticity times the interest rate λR. In first differences the monetary model can be concisely summarized as

$$\Delta s_{0j} = \Delta g_0 - \Delta g_j \tag{2.1}$$

The linear in first difference specification reveals two properties that will prove crucial in the following sections. First, the set of multiple exchange rates Δs_{0j} $(j = 1, \cdots, n)$ all have the fundamental Δg_0 in common. This exposure to shocks in the numéraire currency may be important as illustrated in e.g. Aghion, Bachechetta and Banerjee (2001). For a set of emerging market currencies, they plot the ratio of dollar denominated liabilities to claims with respect to foreign banks in 1997 right before the start of the Asian crisis. Given the high content of dollar denominated debt, most of the emerging market currencies were therefore highly exposed to the same US interest rate fluctuations. Second, (2.1) is linear in the first differences of the composite fundamental g. The US interest rate shocks therefore affected the different emerging market currencies exchange rate returns in a similar linear way. The linear specification also conforms to the linear factor model used in Forbes and Chinn (2003), who show that trade linkages are important transmitters of shocks between countries.[2]

2.1. **portfolios.** Portfolios are by definition linear in the assets and liabilities' returns. Indirectly, portfolios are linear in the risk factors. We will repeatedly use the following example two-asset portfolio to illustrate the implications of our results. Consider the case of syndicated loans. Suppose there are just two projects with random excess returns X and Y. Both returns have the following structure

$$X = \beta_x R + \varepsilon_x$$

and

$$Y = \beta_y R + \varepsilon_y.$$

Here R is the excess return on the market portfolio over the risk free rate and ε_x and ε_y are the idiosyncratic risks (independent from the market risk); beta's indicate how much the projects covary with the market. Below, we often equate the beta to zero and later indicate how to deal with non-zero betas. With non-zero betas, the results become in fact easier rather than more difficult to derive, since dependence is built in from the very beginning. We can allow the projects' returns to be time dependent, see below where we discuss the case of ARCH processes for the risk drivers. For the purpose of the example, we will also assume that ε_x and ε_y are independent. Again, this is not necessary.[3] Imagine that the loans for the projects are underwritten by two investment banks or sold on to two financial intermediaries, like commercial banks or insurance companies. Let bank one hold the portfolio

$$Q = (1-\gamma)X + \gamma Y, \tag{2.2}$$

while the loan portfolio of bank two is

$$W = \gamma X + (1-\gamma)Y. \tag{2.3}$$

Here γ is restricted to be in the interval between zero and one. Note that the correlation between the two portfolios is

$$\rho = 1 - \frac{1-4\gamma(1-\gamma)}{1-2\gamma(1-\gamma)}.$$

Hence for $\gamma \epsilon (1/2, 1)$ the correlation is nonzero.

[2] Note that the monetary model captures the mirror image of the trade account through movements in the capital account.

[3] As long as the dependence between ε_x and ε_y is linear, it can be analyzed within the framework we develop below.

2.1.1. *large portfolios.* Much of what we do below is for the case of arbitrarily sized portfolios. In general the portfolio returns are

$$Q = \sum_{i=1}^{n} \lambda_i X_i$$

and

$$W = \sum_{i=1}^{n} \gamma_i X_i.$$

In these portfolios we can allow for short selling and portfolios can be "unbalanced" in the sense that some assets are not present in all portfolios. To treat the case of unbalanced portfolios, we need at least three assets. In some instances the case of unbalanced portfolios is qualitatively different from the case of balanced portfolios. Unless stated otherwise, the X_i asset returns are linear instruments or risk factors. To deal with the case of non-zero betas or dependent idiosyncratic risks can be analyzed within this multivariate portfolio framework. Take the case of non-zero betas for the example portfolio from the previous subsection. Interpret $X_1 = R$, $X_2 = \varepsilon_x$, $X_3 = \varepsilon_y$ and let $\lambda_1 = \beta_x(1-\gamma)+\beta_y\gamma$, $\lambda_2 = 1-\gamma$, $\lambda_3 = \gamma$, $\gamma_1 = \beta_x\gamma + \beta_y(1-\gamma)$, $\gamma_2 = \gamma$, $\gamma_3 = 1-\gamma$, to study the non-zero betas portfolio. We investigate the role of non-linear derivative instruments separately. In addition we will also investigate the case multiple, ie more than two, banks.

2.2. **discrete and continuous returns.** Depending on the problem at hand, either continuous or discrete returns are used. Some classes of assets, such as in the case of stocks, have almost continuous price formation in time and thus continuous compounding is typically used for such assets. Other assets only trade or payout at discrete instances in time, and hence discrete returns appears more appropriate. Portfolio returns can be obtained by summing the weighted discrete returns; using logarithmic returns this is not possible. Per contrast, aggregation over time works well with continuous returns, whereas discrete returns do not add up. For small price movements, the two concepts of return are close (as a Taylor approximation shows). Let $P(t)$ denote the asset price at time t, and let X be the return. The continuously compounded return is

$$X(t+1) = \log \frac{P(t+1)}{P(t)},$$

where $X \in \mathbb{R}$. Discrete returns are computed as

$$Y(t+1) = \frac{P(t+1)}{P(t)} - 1,$$

where $(Y+1) \in \mathbb{R}^+$. Under continuous compounding the loss return can be as large as can be imagined, but the discrete return can not be worse than -1. This simple lower bound restriction for discrete returns has immediate interesting implications for the tail shape of the multivariate return distribution on the loss side. We investigate the implications for the joint loss distribution both under continuous compounding and discrete compounding. Though most of the time we deal with continuously distributed returns (either under discrete or continuous compounding), we briefly investigate the systemic stability if there are mass points. The motivation for this is twofold. Many of the theoretical crises models employ the Bernoulli distribution. Options and other non-linear instruments may have mass points in their return distribution, whereas the distribution of the underlying asset would not.

Thus we have to investigate both loss distributions with a lower bound on the support and loss distributions with unbounded supports. As we will see below, when we focus on the tail area there are two different classes of distributions to which the distributions with unbounded support can belong. For the class with bounded supports, there are also two classes. Interestingly, one of the two classes is common to both types of distributions. For convenience, we will often work with the distribution of losses, which maps the loss side of the distribution into the positive half axis.

3. Measures of Dependency

We first argue why there is a need for a measure of dependency to reflect systemic risk and why standard concepts like the correlation measure are less suitable for the question at hand. Then we suggest a measure which is close in spirit to the popular VaR measure, but which is suitable at the systems level.

Indeed, why would economists be interested in a measure for systemic risk? Take the banking sector's risk regulatory background as laid down in the recently revamped Basle accords for bank capital holding. The surprising fact is that the entire approach has a predominant micro-prudential orientation, focussing on the risk management practices at individual banks, without much attention to the systemic ramifications (even though the systemic stability is often invoked as a motivation for the necessity of the Basle rules). The banking sector has important externalities within the banking sector and to the rest of the economy. Moreover, an important part of the risks are endogenous to the sector. To give one stark example from the investment industry, recall the popularity and fall from grace of the portfolio insurance technique for managing risk. While evidently prudential from a micro oriented point of view, the technique faltered when all institutions were trying hedge by selling off at the same time on black Monday in October 1987. The fragility of the banking sector is often related to the interbank market where the failure of one bank may impair other banks due to their mutual exposures, causing a domino effect of failures. The banking sector is, moreover, heavily exposed to macro policy risks, such as interest rate policies pursued by the monetary authority, and other macro risk factors. Empirically, the common exposures to macro risk drivers tends to be the dominant source behind the banking sector's instability. For other parts of the financial service sector such as the insurance industry, endogenous risks and domino effects tend to be less important. Insurance is more of an actuarial game against nature and while deposit contracts facilitate bank runs, insurance contracts do not enable similar practices. Thus historically the (re-)insurance industry has been less severely regulated and supervised. Given the specific sensitivity of the banking sector to the macro and endogenous risks, Borio (2003) pleads for giving an explicit role to the macroprudential aspects of bank regulation and supervision on top of the microprudential framework which is now firmly in place through the Basle II accord.

Borio sees the distinction between the micro prudential and macro prudential approaches as follows. The existing micro prudential operational framework focusses on the tail losses (Value at Risk or VaR) of each institution individually. The new macro prudential approach should care about the tail losses of the banking system as a whole. To see what this may amount to, consider the following portfolio examples. Under project allocation I, Bank1 ($B1$) has return exposure X, while Bank2 ($B2$) has sole exposure to Y. In the risk allocation II, the exposures are respectively $Y-X$ and $Y+X$. Assume that either X and Y are independent standard normally distributed, or that they follow independent Student-t distributions with α degrees of freedom. The figures in the Appendix give cross plots for representative samples of these portfolios. Let s be a large VaR level. The following table provides the (approximate) individual and systems failure probabilities. From this table we see that in case the returns are normally distributed, the systems failure probabilities are of smaller order than the individual failure probabilities. Under these circumstances, one might argue that a micro prudential risk supervisory approach suffices. In case the returns are Student distributed, however, the systemic risk is of the same order as the individual failure probabilities. Note that this happens even though the returns are uncorrelated in both cases. But also note that under Allocation II, the Student-t portfolios are not independent (see the figures in the Appendix). When the banking risks exhibit heavy tails like in the case of a Student distribution, one might thus argue there exists a genuine case for adopting a macroprudential approach towards risk supervision.

Distribution	N	S	N	S
Bank Portfolios	$1 - P(B1 \le s, B2 \le s)$		$P(B1 + B2 > 2s)$	
Allocation I	$\sqrt{\frac{2}{\pi}}\frac{1}{s}e^{-s^2/2}$		$\frac{1}{2\sqrt{\pi}}\frac{1}{s}e^{-s^2}$	
X, Y		$2s^{-\alpha}$		$\frac{1}{2^{\alpha-1}}s^{-\alpha}$
Allocation II	$\frac{2}{\sqrt{\pi}}\frac{1}{s}e^{-s^2/4}$		$\frac{1}{\sqrt{2\pi}}\frac{1}{s}e^{-s^2/2}$	
$Y - X, Y + X$		$3s^{-\alpha}$		$s^{-\alpha}$

TABLE 1. Individual and Systemic Failure Probabilities

Motivation for interest in the joint failure probability arises out of concern for the network effects. For example, if there are important cost complementarities within an industry, such as the hub and spokes system in the airline industry, the failure of one company can have important negative feedback effects on other firms and the service level of the whole industry. For the banking industry, the survival of the payment and clearing system is important for the functioning of the entire economy and a single failure can easily implicate other banks. Thus a bank supervisory agency may be imagined to be endowed with the a following two-part welfare function. Out of concern for consumer protection, the agency cares about the individual bank VaR (X_i) levels: $\sum_{i=1}^{n} \frac{1}{n} X_i$.[4] This is the utlitarian part of the welfare function, reflecting current practices under the Basle accords. But in addition, the agency also cares about the viability of the system as a whole. Suppose this system's part is captured by adding a maximin part to the welfare function: $\sum_{i=1}^{n} \frac{1}{n+1} X_i + \frac{1}{n+1} \min_i(X_i)$.[5] To fix ideas, consider the bank portfolio discussed in the subsection 2.1 on portfolios. Thus Bank1 has loss returns $Q = (1-\gamma)X + \gamma Y$, while Bank 2 has loss returns $W = \gamma X + (1-\gamma)Y$, and where $\gamma \epsilon (1/2, 1)$. Suppose the loss returns X and Y are either independently unit exponentially distributed, or are Pareto distributed (with exponent α). The Table 3 provides, to a first

[4]In the example, we weigh the different VaR's equally, but one can allow for different weights if so desired.

[5]Different specifications can be imagined for this system's part, but the analysis here is only to motivate the choice of the dependency measure; which does not rely on the specifics of the welfare function.

order, the utilitarian and the utilitarian-cum-maximin large loss probabilities. In case of the exponential hypothesis, adding the concern for the system to the welfare function is seen to be relative unimportant since $3(2\gamma-1)^{-1}/2t$ tends to zero as the loss level t increases. In case of the Pareto hypothesis the ratio of the two probabilities is $\left(\frac{4}{3}-\frac{2}{3}\gamma\right)^{\alpha}$, which is in the interval $([\frac{2}{3}]^{\alpha},1)$. Hence, in the Pareto case the two parts of the welfare function are of the same order of importance. Note that different weights on the utilitarian and maximin parts will change the probability ratios quantitatively, but not qualitatively; that is, under the exponential distribution the ratio tends to zero, while under Pareto case it is bounded away from zero.

Welfare functions Distributions	$P(Q+W>2t)$	$P(Q+W+\min(Q,W)>3t)$
Exponential	$2te^{-2t}$	$\frac{3}{2\gamma-1}e^{-2t}$
Pareto	$2^{1-\alpha}t^{-\alpha}$	$2(\frac{2-\gamma}{3})^{\alpha}t^{-\alpha}$

TABLE 2. Utilitarian and Utilitarian cum Maximin Failure Probabilities

The qualitative difference between the two cases is due to the different properties of the joint distribution functions. Since in practice we do not know the specifics of the welfare function (or cost complementarities), while we may be able to deduce the qualitative properties of the joint distribution function (through econometric estimation) and as one wants to circumvent arbitrariness, we propose to use the loss probabilities for constructing a measure of dependency. This measure will then be independent from the specific weight given to the systemic aspect, but has the advantage that it provides a non disputable benchmark (everyone can agree that 22 degrees Celsius is more than 16, while some may find 22 warm and others find this chilly). Using the loss probabilities will give a measure which is close in spirit to the VaR measure, except that it is stated inversely in terms of large loss probabilities, rather than loss levels. This has the advantage that it allows one to specify different loss levels for different banks, while still providing a single index number for the risk of the entire system. It also has the benefit that we do not have to say which loss level actually constitutes a crisis. Supervisors would probably be hard pressed to agree to on such a specific numerical value. But before we discuss our measure in detail, we first briefly review some other measures of dependency.

3.1. **the correlation measure.** Perhaps the most commonly used measure of (linear) dependency is the coefficient of correlation ρ. The means, variances and the correlation coefficient of a pair of random variables completely characterize the bivariate normal distribution. For the bivariate normal distribution with means zero and unit variances the regression curve is governed by ρ :

$$E[Y|X = x] = \rho x.$$

The measure is easy to calculate and can be readily extended to a multivariate setting.

One must ask, however, how well ρ captures the dependency if it is unknown whether the data are normally distributed or not. Specifically, the question is whether ρ adequately captures the interdependency at crisis levels. Boyer, Gibson, and Loretan (1997) note for example that the idea of conditioning on a high loss level in calculating ρ, may give the false appearance of the market speak of increased correlation during times of crisis. The empirical literature finds little support for normality of the return distribution of many asset classes, and while the data may be dependent, the correlation coefficient can be zero (see the figures in the Appendix). For example, in the following bivariate toy conditional volatility model motivated by Engle's (1982) univaraite ARCH model, the returns X and Y exhibit the characteristic heavy tail property, the clustering of volatility, are interdependent, but nevertheless uncorrelated:

$$X_t = N_t H_t, \quad N_t \text{ iid } N(0,1),$$

$$Y_t = M_t H_t, \quad M_t \text{ iid } N(0,1),$$

$$H_t = w + \beta(X_{t-1}^2 + Y_{t-1}^2), 0 \leq \beta < 1/2.$$

Here $\rho_{t-k}(X_t, Y_t) = 0, k = 1, 2...$ even though X_t and Y_t are not independent since they are driven by the same conditional variance function H_t.[6] Thus ρ does not capture the dependency that is in the data. Another concern is that the correlation concept requires that the first two moments are bounded. This is at least of some concern in the non-life branch of the insurance industry, where the loss distribution often appears to fail to have a second moment. Lastly, economists evaluating investments within expected utility theory frameworks are not so much interested in the correlation measure itself; they rather have an interest

[6]Note that regarding volatility spill-overs the correlations measure gives $\rho_{t-1}(X_t^2, Y_t^2) = 0$, but $\rho_{t-k}(X_t^2, Y_t^2) \neq 0$ for $k > 1$.

in the trade-offs between risk measured as a probability and the associated gains or losses, which are the quantiles of the return distribution. As such the correlation is only an intermediate step in the calculation of this trade-off between quantile and probability. Therefore we like to turn to measures which are not conditioned on a particular multivariate distribution and which directly reflect the probabilities and associated crash levels.

3.2. **copulas.** Copulas are rapidly gaining in popularity as a means for describing the dependency. All information concerning a pair of random variables is contained in their joint distribution function $F(x, y)$. Thus this function contains both information regarding the marginals and their dependency structure. A copula $C(.,.)$ isolates the dependency structure from $F(x, y)$ by transforming the arguments via the marginals

$$C(F_x(x), F_y(y)) = F(x, y).$$

Hence the shape of $F_x(x)$ and $F_y(y)$ is filtered out in the copula. The copula entirely determines the dependency structure. Recently certain copulas other than the Gaussian, have become popular in finance. The copulas at the two ends of the copula spectrum are the independent copula $C(x, y) = xy$ and the copula for perfect dependence $C(x, y) = \min(x, y)$. We review a few of the most commonly used copulas: Longin and Solnik (2001) used the Logistic copula to estimate the dependency between equity markets ($\beta = 1$ gives the independent copula)

$$C(x, y; \beta) = \exp[-\{(-\ln x)^{1/\beta} + (-\ln y)^{1/\beta}\}^{\beta}],\ 0 < \beta \leq 1.$$

The Morgenstern copula reads ($\delta = 0$ gives the independent copula)

$$C(x, y; \delta) = xy[1 + \delta(1 - x)(1 - y)],\ \ -1 \leq \delta \leq 1;$$

the Bivariate Pareto copula is

$$C(x, y; \alpha) = x + y - 1 + [(\frac{1}{1 - x})^{1/\alpha} + (\frac{1}{1 - y})^{1/\alpha} - 1]^{-\alpha},\ 0 < \alpha;$$

and the Plackett copula for $\theta \neq 1$ reads

$$C(x, y; \theta) =$$

$$\frac{1}{2\left(\theta - 1\right)}[1 + \left(\theta - 1\right)\left(x + y\right) - \sqrt{[1 + \left(\theta - 1\right)\left(x + y\right)]^2 - 4xy\theta\left(\theta - 1\right)}].$$

The popularity of the copula approach partly derives from the fact that they lend themselves easily to parametric estimation. Given our objective to uncover the dependency in the tail area, however, the parametric copula approach shares the same problem with parametric distribution based approaches, like the Gaussian approach, in that it does not necessarily do justice to the behavior in the tail area. Moreover, as of to date there does not exist an economic motivation for choosing one copula over the other. We hope to shed some light on both these issues below. For supervisors and regulators, using a function rather an index, the disadvantage is that a function is in general less succinct, and hence may not be acceptable as a summary measure for dependency. It can also be less robust if one lacks part of the necessary information to construct the function.

3.3. **co-crash probabilities.** In the introduction to this section we argued extensively that the macro prudential concern is a heavy loss in one bank which goes hand in hand with a heavy loss of an other bank, creating a breakdown of the financial system. For several reasons we tried to argue that this concern is best reflected by the joint failure probability. Since we are interested in the extreme linkage probabilities we will directly evaluate these probabilities, bypassing the correlation concept. More specifically, one asks given that $Y > t$, what is the probability that $X > s$, where X and Y stand for asset returns and t, s are high loss levels, or vice versa. We propose to use the probability measure which conditions on any market crash, without indicating the specific market. This is the linkage measure[7]

$$\frac{P\{X > s\}}{1 - P\{X \leq s, Y \leq s\}} + \frac{P\{Y > s\}}{1 - P\{X \leq s, Y \leq s\}}$$

first proposed in Huang [22] and implemented empirically by Hartmann et al. (2004). The linkage measure, even though it is the sum of two conditional probabilities, reflects the expected number of market crashes given that least one market has collapsed. To see this, let κ_s denote the number of simultaneously crashing asset markets, i.e., returns exceeding s, and write the conditionally expected number of asset market crashes given a collapse in at least one asset market as $E\{\kappa_s|\kappa_s \geq 1\}$. Then

$$\begin{aligned} E\{\kappa_s|\kappa_s \geq 1\} &= E\{\mathbf{1}_{X>s} + \mathbf{1}_{Y>s}|\kappa_s \geq 1\} \\ &= E\{\mathbf{1}_{X>s}|\kappa_s \geq 1\} + E\{\mathbf{1}_{Y>s}|\kappa_s \geq 1\} \\ &= P\{X > s|\kappa_s \geq 1\} + P\{Y > s|\kappa_s \geq 1\} \\ &= \frac{P\{X > s\} + P\{Y > s\}}{P\{X > s \text{ or } Y > s\}} \end{aligned} \tag{3.1}$$

[7]We take the two quantiles on which we condition equal to s, but this is by no means necessary. If two banks have quite different levels of capital, one may want to take the loss return thresholds with a systemic impact different.

Compare this expression with another quantity which is sometimes used to measure dependence in the tail:

$$\theta_s := \frac{P\{X > s \text{ and } Y > s\}}{P\{X > s \text{ or } Y > s\}}. \tag{3.2}$$

We have

$$E\{\kappa_s | \kappa_s \geq 1\} = 1 + \theta_s.$$

The advantage of the expectation measure is that it can be easily extended to higher dimensions, while the θ_s is more cumbersome and less revealing, see below. Also note that the structure of the joint failure probability $P\{X > s \text{ and } Y > s\}$ in the numerator determined the orders of magnitudes of the entries in the tables 3 and 3.

Suppose now that X and Y have the same probability distribution. Then, under extreme value conditions, the limit

$$\kappa = \lim_{s\to\infty} E\{\kappa_s | \kappa_s \geq 1\} = 1 + \lim_{s\to\infty} \theta_s$$

exists. This limit can be used as an indicator for the amount of dependence in the tail between X and Y. One reason to take the limit, rather than using a finite loss level s, is that economics does not say what the critical level is at which systemic failure sets in. Taking the limit thereby removes some arbitrariness. At the same time, the limit is still indicative about what happens at high but finite loss levels. Note that

$$\lim_{s\to\infty} E\{\kappa_s | \kappa_s \geq 1\} = 1$$

can be interpreted as asymptotic independence. This obtains in particular when $C(x, y) = xy$. In case

$$\lim_{s\to\infty} E\{\kappa_s | \kappa_s \geq 1\} = 2,$$

this is interpreted as asymptotic maximal dependence. In particular, if $C(x, x) = x$, this limit applies. Hence

$$H := \lim_{s\to\infty} E\{\kappa_s | \kappa_s \geq 1\} - 1 \tag{3.3}$$

is a number between 0 and 1. It can be used as a measure of asymptotic dependence in the tail in a way analogous to the correlation coefficient for the tail distribution. However there is no direct connection: even for a normal distribution with coefficient of correlation $r \neq 0$, we find $H = 0$.

In the higher dimensional situation (with $d > 2$ random variables) H can be defined in a completely analogous way:

$$H := \frac{\lim_{s\to\infty} E\{\kappa_s | \kappa_s \geq 1\} - 1}{d - 1}$$

and the interpretation is the same. Note that the straightforward generalization of (3.2):

$$\lim_{s\to\infty}\theta_s := \lim_{s\to\infty}\frac{P\{X_1 > s, \ldots, X_d > s\}}{P\{X_1 > s \textit{ or } \ldots X_d > s\}}$$

is not directly linked to H and that $\lim_{s\to\infty}\theta_s$ can be zero even if the random variables are not jointly independent in the tail.

As observed above, for any normal distribution and many other distributions, we find $H = 0$. Even in the case of asymptotic independence one can make a distinction between probability distributions which exhibit more and less dependence by applying a finer scale in the framework of extreme value theory. In the simple case the domain of attraction condition can be written (cf. Resnick, 1987)

$$\lim_{t\to\infty} t\{1 - F(tx, ty)\} = -\log G(x, y) \tag{3.4}$$

for $x, y > 0$ where F is the initial distribution and G the limit distribution. Assume a 'second order' or 'speed of convergence' condition: Suppose there exists a positive function A and a limit function H such that

$$\lim_{t\to\infty}\frac{t(1 - F(tx, ty)) + \log G(x, y)}{A(t)} \to H(x, y) \tag{3.5}$$

for $0 < x, y \le \infty$ (we include $x = \infty$ and $y = \infty$ since otherwise we would not control the marginal distributions). It can be proved that A is regularly varying with index $\rho \le 0$. Now in case of asymptotic independence

$$G(x, y) = G(x, \infty)G(\infty, y) = e^{-\frac{1}{x}-\frac{1}{y}}. \tag{3.6}$$

Also (3.5) with $y = \infty$ or $x = \infty$ entails

$$\begin{aligned}&\lim_{t\to\infty}\frac{t(1 - F(tx, \infty)) + \log G(x, \infty)}{A(t)} \to H(x, \infty)\\ &\lim_{t\to\infty}\frac{t(1 - F(\infty, ty)) + \log G(\infty, y)}{A(t)} \to H(\infty, y).\end{aligned} \tag{3.7}$$

Then, by combining (3.5), (3.6) and (3.7), it follows that

$$\frac{tP(X > tx, Y > ty)}{A(t)} \to H(x, \infty) + H(\infty, y) - H(x, y). \tag{3.8}$$

Now according to (3.4), $P\{X > tx \textit{ or } Y > ty\}$ is asymptotically of order t^{-1}, i.e., regularly varying with index -1, whereas according to (3.8) $P\{X > tx \textit{ and } Y > ty\}$ is asymptotically of order $t^{-1}A(t)$, i.e.

regularly varying with index $\rho - 1$. Ledford and Tawn [24] introduced the parameter η defined as

$$\eta = \frac{1}{1-\rho} \in [0,1]$$

as a measure to distinguish between asymptotically independent distributions. Note that if $H > 0$, then $\eta = 1$. Note also for a normal distribution $\eta = (1+r)/2$, with r the coefficient of correlation. Note that if X and Y are independent, $\eta = 1/2$, but the converse does not hold.

If we combine the H scale and the η scale, we can define the Fragility Index FI:

$$FI = \begin{cases} \lim_{s\to\infty} E\{\kappa_s | \kappa_s \geq 1\} & \text{if } H > 0 \\ \frac{1}{2} \lim_{s\to\infty} \frac{\log P\{X > s\} + \log P\{Y > s\}}{\log P\{X > s, Y > s\}} & \text{if } H = 0. \end{cases}$$

We will say that the financial system is *strongly fragile* if $FI > 1$, while the system is only *weakly fragile* if $FI \epsilon [0,1]$.

Finally we discuss the extension to higher dimensional space of Ledford and Tawn's η. We start from the extension of (3.5):

$$\lim_{t\to\infty} \frac{tP\{X > tx \text{ or } Y > ty \text{ or } Z > tz\} - (\frac{1}{x} + \frac{1}{y} + \frac{1}{z})}{A(t)} = H(x,y,z).$$

Now (cf. (3.7) and the reasoning thereafter)

$$\begin{aligned} & P\{X > tx \text{ or } Y > ty \text{ or } Z > tz\} - P\{X > tx\} \\ & \quad - P\{Y > ty\} - P\{Z > tz\} = \\ & \quad P\{X > tx, Y > ty\} + P\{X > tx, Z > tz\} \\ & \quad + P\{Y > ty, Z > tz\} - P\{X > tx, Y > ty, Z > tz\}. \end{aligned} \tag{3.9}$$

Suppose that all two-dimensional marginal distributions satisfy (3.8), i.e.

$$\lim_{t\to\infty} \frac{tP(X > tx, Y > ty)}{A(t)} \to H(x,\infty,\infty) + H(\infty,y,\infty) - H(x,y,\infty)$$

$$\lim_{t\to\infty} \frac{tP(Y > ty, Z > tz)}{A(t)} \to H(\infty,y,\infty) + H(\infty,\infty,z) - H(\infty,y,z)$$

$$\lim_{t\to\infty} \frac{tP(X > tx, Z > tz)}{A(t)} \to H(\infty,x,\infty) + H(\infty,\infty,z) - H(x,\infty,z)$$

Then (cf. (3.9)) it remains to deal with $P\{X > tx, Y > ty, Z > tz\}$. It is possible that this probability is of the same order as the two-dimensional distributions, i.e. of order $t^{-1}A(t)$, or this probability is of lower order and then we are dealing with a new parameter, smaller than the η that comes from A, determining the joint exceedance of all three variables.

4. Weak and Strong Financial Fragility

In view of the above linear bank portfolio framework to determine the amount of fragility of a network, we need a theory about the dependence between weighted sum of random variables (asset returns) in the tail areas. It follows from the theory below that if for example asset returns are normally distributed, the financial fragility is weak, while if the returns are uniform or Student distributed, the system can become strongly fragile. The section first deals with discrete returns, ie with distributions that have an endpoint, then it turns to continuous returns, and lastly briefly investigates the effects of mass points. The part on continuous returns is divided into a part with light and with heavy tails. The cases of discrete returns and continuous returns can be fully characterized; the case of continuous returns with light tails is only partially treated.

More specifically, as we have seen, the fragility of the system is linked to the joint tail behavior of linear portfolio combinations $R = \lambda_1 X_1 + \lambda_2 X_2$ and $Z = \mu_1 X_1 + \mu_2 X_2$. It is assumed that X_1 and X_2 are i.i.d. We are interested in high values of the vector (R, Z) and in particular in the dependence between R and Z in the tail area. We consider the right tails and assume that the distribution of X_1 and X_2 is in the domain of attraction of an extreme value distribution G_γ.[8] In order to be able to calculate the fragility we need to find the joint tail behavior of R and Z for different values of γ. For the cases $\gamma > 0$ and $\gamma < 0$ we give a complete characterization; in case $\gamma = 0$ we only have partial results. Throughout we will use the notation $\alpha =: 1/\gamma$ if $\gamma \neq 0$.

4.1. weak and strong financial fragility and discrete returns, $\gamma > 0$.

We start by showing that the class of distributions in the domain of attraction of the Weibull limit law is closed under addition.

4.1.1. *Closure under addition.* Let $F(x)$ be a loss distribution with bounded support $[0, a]$, hence $F(a) = 1$, $0 < a < \infty$. Suppose $F(x)$ is in the domain of attraction of a Weibull extreme value distribution. It is shown that convolutions remain in this domain of attraction.

[8] Where $\gamma > 0$ refers to the Weibull limit law, $\gamma < 0$ is the Frechet case, and $\gamma = 0$ represents the Gumbel limit law. The extreme value theorem holds that the limit law for the minimum (maximum) is either one of these dsitributions.

Define the "upper tail distribution" $H(x)$ as

$$H(x) \equiv 1 - F(-x + a)$$

Introduce the Laplace transform $\widetilde{H}(y)$

$$\widetilde{H}(y) \equiv \int_0^\infty y e^{-yx} H(x) dx. \tag{4.1}$$

Hence for $t > 0$ by a transformation of variable

$$\frac{\widetilde{H}(y/t)}{H(t)} = y \int_0^\infty e^{-yx} \frac{H(tx)}{H(t)} dx.$$

We will use the following result.

Lemma 1. *Suppose $H(x)$ is in the domain of attraction of the Weibull extreme value distribution, i.e.*

$$\lim_{t \downarrow 0} H(tx)/H(t) = x^\alpha, \alpha > 0 \tag{4.2}$$

Then the two convolution H^{2*} of $H(x)$ is again in the domain of attraction of the Weibull extreme value distribution and satisfies

$$\lim_{t \downarrow 0} \frac{H^{2*}(tx)}{H^{2*}(t)} = x^{2\alpha}.$$

Proof. By (4.2) and a transformation of variable

$$\begin{aligned} \lim_{t \downarrow 0} \frac{\widetilde{H}(y/t)}{H(t)} &= y \int_0^\infty e^{-yx} x^\alpha dx \\ &= y^{-a} \int_0^\infty x^\alpha e^{-x} dx \\ &= y^{-\alpha} \Gamma(1 + \alpha) \qquad (4.3) \end{aligned}$$

Hence the Laplace transform $\widetilde{H}$ varies regularly at infinity with tail index $-\alpha$ whenever H varies regularly at zero with index α. By the convolution theorem for Laplace transforms

$$\widetilde{H}^{2*}(y) = \left(\widetilde{H}(y)\right)^2.$$

Hence, $\widetilde{H}^{2*}(y)$ varies regularly at infinity with index -2α. By (4.3) this implies that the two convolution $H^{2*}(x)$ varies regularly at zero with index 2α. □

Thus the class is closed under addition, but the index of regular variation changes.

Remark 1. *This convolution implies that if portfolios contain an equal number of assets (with returns in the class), the portfolios have the same index of regular variation, while if two portfolios differ with respect to the number of assets, their indices differ. As we will see, this determines whether or not the portfolios are asymptotically dependent or independent.*

4.1.2. *asymptotic dependence.* By assumption, we have

$$\lim_{n\to\infty} nP\left\{X_1 > xU(n)\right\} = x^{-\alpha} \text{ for } x > 0,\ i = 1,2 \tag{4.4}$$

where U is the inverse function of the distribution function of X_i at the point $1 - 1/x$.

In view of (4.25) we have

$$\lim_{n\to\infty} nP\left\{\lambda_1 X_1 + \lambda_2 X_2 > xU(n) \text{ or } \mu_1 X_1 + \mu_2 X_2 > yU(n)\right\}$$
$$= \left(\frac{x}{\lambda_1} \wedge \frac{y}{\mu_1}\right)^{-\alpha} + \left(\frac{x}{\lambda_2} \wedge \frac{y}{\mu_2}\right)^{-\alpha}.$$

It follows (cf Ferreira and de Haan ch. 6) that the vector (X_1, X_2) is in the domain of attraction of a (symmetric) extreme value distribution with marginal extreme value index γ and discrete spectral measure concentrated on two points in the interior of its range. The fragility can be determined as in example (2) of section 6.

4.1.3. *Case 2.* $\gamma < 0$. For simplicity of writing we assume that the right end point of the distribution function (which must be finite) is zero. Then for $x > 0$, $i = 1, 2$

$$\lim_{n\to\infty} nP\left\{-X_i < xU(n)\right\} = x^{-\alpha}$$

Hence as $n \to \infty$ for $x, y > 0$

$$\begin{aligned} &n^2 P\left\{-X_1 < xU(n) \text{ and } -X_2 < yU(n)\right\} = \\ &\quad nP\{-X_1 < xU(n)\}nP\{-X_2 < yU(n)\} \to \\ &\quad (xy)^{-\alpha} = \alpha^2 \int_0^y \int_0^x s^{-\alpha-1} t^{-\alpha-1} ds dt. \end{aligned} \tag{4.5}$$

We claim that for $x, y, \lambda_1, \lambda_2, \mu_1, \mu_2 > 0$

(4.6)
$$\lim_{n\to\infty} n^2 P\{-(\lambda_1 X_1 + \lambda_2 X_2) < U(n)x \text{ or } -(\mu_1 X_1 + \mu_2 X_2) < U(n)y\}$$
$$= \alpha^2 \int\int_S s^{-\alpha-1} t^{-\alpha-1} ds dt,$$

where $S = \{(s,t) : \lambda_1 s + \lambda_2 t \le x, \mu_1 s + \mu_2 t \le y, s > 0, t > 0\}$. Now (4.5) entails

$$\begin{aligned} &\lim_{n\to\infty} n^2 P\{x_1 U(n) \le -X_1 < x_2 U(n), y_1 U(n) \le -X_2 < y_2 U(n)\} \\ &\quad = \alpha^2 \int_{y_1}^{y_2} \int_{x_1}^{x_2} s^{\alpha-1} t^{\alpha-1} ds dt, \end{aligned} \tag{4.7}$$

for $0 \le x_1 < x_2 < \infty, 0 \le y_1 < y_2 < \infty$, i.e. we have convergence for rectangles and for finite unions of rectangles.

It clearly suffices for the proof of (4.6) to give a proof for sets A_m and B_m such that $A_m \subset S \subset B_m$ and $B_m \backslash A_m \downarrow \emptyset, m \to \infty$. Note that in case $\lambda_1\mu_2 \ne \lambda_2\mu_1$ and $x/y \in (\lambda_1/\mu_1, \lambda_2/\mu_2)$, the boundary of S consists of 4 line segments. The vertices are $(0,0)$, $(a,0) := (\min(\frac{x}{\lambda_1}, \frac{y}{\mu_1}), 0)$, $(0,b) := (0, \min(\frac{x}{\lambda_2}, \frac{y}{\mu_2}))$ and

$$(s_0, t_0) := \frac{1}{\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix}} \left(\begin{vmatrix} x & \lambda_2 \\ y & \mu_2 \end{vmatrix}, - \begin{vmatrix} x & \lambda_1 \\ y & \mu_1 \end{vmatrix}. \right) \tag{4.8}$$

We concentrate on the subarea S_1 with vertices $(0,0), (0,b), (s_0, t_0)$ and $(s_0, 0)$. Define for $i = 1, \ldots, m$

$$s_i := \frac{(m-i)s_0}{m} \text{ and } t_i := \frac{t_0 - b}{s_0} s_i + b$$

and the sets

$$L_i := (s,t) : s_i \le s < s_{i-1}, 0 \le t \le t_i$$

and

$$U_i := (s,t) : s_i \le s < s_{i-1}, 0 \le t \le t_{i+1}.$$

Then $\cup_{i=1}^m L_i \subset S_1 \subset \cup_{i=1}^m U_i$ and by (4.7)

$$\lim_{n\to\infty} n^2 P\{U(n)^{-1}(X_1, X_2) \in \cup_{i=1}^m L_i\} = \alpha^2 \int\int_{(s,t)\in\cup_{i=1}^m L_i} s^{-\alpha-1} t^{-\alpha-1} ds dt$$

and

$$\lim_{n\to\infty} n^2 P\{U(n)^{-1}(X_1, X_2) \in \cup_{i=1}^m U_i\} = \alpha^2 \int\int_{(s,t)\in\cup_{i=1}^m U_i} s^{-\alpha-1} t^{-\alpha-1} ds dt.$$

Since clearly $\cup_{i=1}^m U_i \backslash \cup_{i=1}^m L_i \to \emptyset$ as $m \to \infty$, we have proved (4.6).

Let us now simplify the integral at the right-hand side of (4.6). We start with the subarea S_1.

$$\begin{aligned}
&\alpha^2 \int\int_{(s,t)\in S_1} s^{-\alpha-1}t^{-\alpha-1}dsdt \\
&= \alpha^2 \int_0^{s_0}\int_0^{b-\frac{b-t_0}{s_0}s} t^{-\alpha-1}dts^{-\alpha-1}ds \\
&= -\alpha\int_0^{s_0} s^{-\alpha-1}(b-\frac{b-t_0}{s_0}s)^{-\alpha}ds \\
&= -\alpha b^{-\alpha}\int_0^{s_0} s^{-\alpha-1}(1-\frac{b-t_0}{bs_0}s)^{-\alpha}ds \\
&= -\alpha b^{-\alpha}\left(\frac{bs_0}{b-t_0}\right)^{-\alpha}\int_0^{1-t_0/b} u^{-\alpha-1}(1-u)^{-\alpha}du
\end{aligned}$$

Similarly, the integral over the mirror area S_2 with vertices $(0,0)$, $(0,t_0)$, (s_0,t_0), $(a,0)$ is

$$-\alpha a^{-\alpha}(\frac{at_0}{a-s_0})^{-\alpha}\int_0^{1-s_0/a} u^{-\alpha-1}(1-u)^{-\alpha}du.$$

Next note that the sum of the integrals over S_1 and S_2 is equal to the integral over S minus the integral over $[0,s_0]\times[0,t_0]$, which is $(s_0t_0)^{-\alpha}$.

It follows that the right-hand side of (4.6) is

$$\begin{aligned}
&-\alpha b^{-\alpha}\left(\frac{bs_0}{b-t_0}\right)^{-\alpha}\int_0^{1-t_0/b} u^{-\alpha-1}(1-u)^{-\alpha}du \\
&-\alpha a^{-\alpha}(\frac{at_0}{a-s_0})^{-\alpha}\int_0^{1-s_0/a} u^{-\alpha-1}(1-u)^{-\alpha}du-(s_0t_0)^{-\alpha}.
\end{aligned} \tag{4.9}$$

Note that for $p,q>0$, $x\in(0,1)$, $B_x(p,q):=\int_0^x t^{p-1}(1-t)^{q-1}dt$ is the incomplete beta function, see e.g. Abramowitz and Stegun [1], p. 263.

Reformulation of both sides in (4.6) shows that we have proved the following theorem.

Theorem 1. *With the notation given in the introduction of the section, in case $\gamma<0$, it follows that the vector (R,Z) is in the bivariate domain of attraction of an extreme value distribution, i.e. for $x,y>0$*

$$\begin{aligned}
&P^n\{\lambda_1X_1+\lambda_2X_2>-xU(\sqrt{n}),\mu_1X_1+\mu_2X_2>-yU(\sqrt{n})\} \\
&\to exp\{-\alpha^2\int_S(st)^{-\alpha-1}dsdt\},\ (n\to\infty)
\end{aligned} \tag{4.10}$$

where $S = \{(s,t) : \lambda_1 s + \lambda_2 t \le x, \mu_1 s + \mu_2 t \le y, s > 0, t > 0\}$ *so that the logarithm of the right-hand side equals minus (4.9) with* (s_0, t_0) *as in (4.8).*

So we conclude that $(\lambda_1 X_1 + \lambda_2 X_2,\ \mu_1 X_1 + \mu_2 X_2)$ is in the domain of attraction of a multivariate symmetric EVT distribution function, with extreme value index 2γ and non-discrete spectral measure.

Now we determine the strength of the fragility of two portfolios $R = (1-\gamma)X_1 + \gamma X_2$ and $Z = \gamma X_1 + (1-\gamma)X_2$, where $\gamma \in (0,1), \gamma \ne \frac{1}{2}$, and X_1, X_2 are as in Theorem 4. Note that in this case we take limits as $s \uparrow 0$. Similar to (4.3) we have

$$\kappa = \lim_{s \uparrow 0} E\{\kappa_s | \kappa_s \ge 1\} = \lim_{s\uparrow 0} \frac{P\{R > s\} + P\{Z > s\}}{P\{R > s \text{ or } Z > s\}}$$

First suppose $\frac{1}{2} < \gamma < 1$. In order to evaluate κ we use (4.6) with $\lambda_1 = \mu_2 = 1-\gamma, \lambda_2 = \mu_1 = \gamma$ and $x = y = 1$ to find that

(4.12)
$$n^2 P\{-((1-\gamma)X_1 + \gamma X_2) < U(n) \text{ or } -(\gamma X_1 + (1-\gamma)X_2) < U(n)\}$$
$$\to -2\alpha[\gamma(1-\gamma)]^\alpha \int_0^{1-\gamma} s^{-\alpha-1}(1-s)^{-\alpha} ds - 1.$$

For the marginal distribution we find similarly

$$n^2 P\{-((1-\gamma)X_1 + \gamma X_2) < U(n)\}$$
$$\to \alpha^2 \int_0^{1/\gamma} \int_0^{\frac{1-\gamma t}{1-\gamma}} (st)^{-\alpha-1} ds dt = \frac{\Gamma(1-\alpha)^2[\gamma(1-\gamma)]^\alpha}{\Gamma(1-2\alpha)} \tag{4.13}$$

and combining the results we find

$$\kappa = \frac{2\Gamma(1-\alpha)^2[\gamma(1-\gamma)]^\alpha}{\Gamma(1-2\alpha)\{-2\alpha[\gamma(1-\gamma)]^\alpha \int_0^{1-\gamma} s^{-\alpha-1}(1-s)^{-\alpha} ds - 1\}}.$$

This completes the proof in case $\gamma \in (\frac{1}{2}, 1)$. We omit the proof in case $\gamma \in (0, \frac{1}{2})$ which is similar.

Example 1. *Consider the portfolios* $Q = (1-\gamma)X + \gamma Y$, *and* $W = \gamma X + (1-\gamma)Y$. *Suppose that the marginal loss distributions of* X *and* Y *are uniformly distributed on* $[0,1]$. *By computing the areas above the portfolio lines in the upper right hand corner of the unit square, one readily finds*

$$P\{Q > t\} = P\{W > t\} = \frac{1}{2}\frac{1}{\gamma(1-\gamma)}(1-t)^2$$

and

$$P\{Q > t, W > t\} = \frac{1}{\gamma}(1-t)^2 .$$

Thus

$$\lim_{s\uparrow 1} E\{\kappa|\kappa \geq 1, s\} = \frac{1}{1 - \frac{\frac{1}{\gamma}(1-t)^2}{\frac{1}{\gamma(1-\gamma)}(1-t)^2}} = \frac{1}{\gamma} > 1$$

as $\gamma \epsilon (1/2, 1)$*. It follows that the two portfolios are asymptotically dependent and* FI$=1/\gamma - 1$*. Alternatively, suppose the portfolio* W *is changed into* $W = X$*. In that case,* $P\{W > t\} = 1 - t$*, while* $P\{Q > t\}$ *remains as before. The joint loss probability becomes*

$$P\{Q > t, W > t\} = \frac{3-2\gamma}{2(1-\gamma)}(1-t)^2 .$$

In this case $\lim_{s\uparrow 1} E\{\kappa|\kappa \geq 1, s\} = 1$*, as* $P\{W > t\}$ *is of larger order than the other two probabilities, recall the remark 1 at the beginning of the subsection. Thus the portfolios are asymptotically independent. One also calculates that the* FI *in this case is* $3/4$ *(weak version).*

4.2. **continuous returns with light tails,** $\gamma = 0$**.** We have results for the normal distribution, the exponential distribution and the subexponential distributions which are in the domain of attraction of the Gumbel law. The entire class of subexponentials is treated in the next subsection. The normal case is well know and was first proved by Sibuya (1960). The exponential case is treated in De Vries (2005).

4.3. **continuous returns with heavy tails.** A well known class of heavy tailed distributions is the class of subexponential distributions S. This class is defined by the property

$$P(X_1 + X_2 > x) \sim 2P(X_1 > x) \text{ as } x \to \infty, \tag{4.14}$$

where X_1, X_2 are i.i.d. random variables. The random variables in the definition above are often supposed to be positive, but we will not assume this here since we are dealing with log returns.

A subclass of S are the distribution functions F which have a first order term similar to the Pareto distribution, i.e.

$$F(s) = 1 - s^{-\alpha}L(s) \quad \text{as } s \to \infty, \tag{4.15}$$

where $L(s)$ is a slowly varying function such that

$$\lim_{t\to\infty} \frac{L(ts)}{L(t)} = 1, \quad s > 0. \tag{4.16}$$

It can be easily shown that conditions (4.15)-(4.16) are equivalent to

$$\lim_{t\to\infty} \frac{\overline{F}(ts)}{\overline{F}(t)} = s^{-\alpha}, \quad \alpha > 0,\ s > 0, \tag{4.17}$$

i.e., the tail of the distribution function $\overline{F} := 1 - F$ varies regularly at infinity. The tail index α can be interpreted as the number of bounded distributional moments. And as not all moments are bounded, we speak of heavy tails. Distributions like the Student-t, F-distribution, Burr distribution, sum-stable distributions with unbounded variance all fall into this class. It can be shown that the unconditional distribution of the ARCH and GARCH processes belongs to this class, see De Haan et al. (1989) for a proof. The Student-t distributions are often used in the empirical modelling of the unconditional returns, while GARCH processes are extremely popular conditional models.

Apart from the distributions functions with a regularly varying tail, the class S contains distributions like the lognormal and the Weibull distribution with parameter less than 1. The class S is of importance in ruin theory in insurance, queueing theory and other areas of applied probability. For applications the reader is referred to Asmussen [3], Embrechts et al. [11] or Rolski et al. [31].

In the sequel we will use the inclusion $S \subset L$, where L is the class of long-tailed distributions. A distribution function F is long tailed if it has infinite upper endpoint and $\overline{F} \in L$, i.e. if $\overline{F}(x+a) \sim \overline{F}(x)$ as $x \to \infty$ (for $a \in \mathbb{R}$). In this case convergence is uniform on compact subsets of $\mathbb{R}$.

The class of subexponential distributions is connected with functions of dominated variation as well: the inclusion $L \cap D \subset S$. See Goldie [19]. We write $F \in D$ to denote that the tail function $\overline{F}$ is of dominated variation, i.e. if $\limsup_{x\to\infty} \overline{F}(ax)/\overline{F}(x) < \infty$ for $a < 1$. Note that the implication $L \cap D \subset S$ ensures that distributions functions F with a regularly varying tail function $\overline{F}$ are subexponential. This observation allows us to derive the joint asymptotic distribution for linear combinations of random variables with a regularly varying tail in an indirect way using subexponentiality.

Relation (4.14) above is equivalent to

$$P(X_1 + \ldots + X_n > x) \sim nP(X_1 > x) \ as\ x \to \infty$$

and, since $P(\vee_i X_i > x) \sim nP(X_1 > x)$, the property of subexponentiality can be reformulated as

$$P(\vee_{i=1}^n X_i > x) \sim P(\textstyle\sum_{i=1}^n X_i > x) \ as\ x \to \infty.$$

The above property is called max-sum equivalence. The random variables X, Y with distribution function $F(G)$ are max-sum equivalent (notation $F \sim_M G$) if

$$P(X+Y>x) \sim P(X \vee Y > x) \; as \; x \to \infty, \tag{4.18}$$

equivalently if $P(X+Y>x) \sim P(X>x)+P(Y>x)$. It follows that $F \in S$ if and only if $F \sim_M F$.

The class S is not closed under convolution, see Leslie [25]. However, for $F, G \in S$, the convolution $F * G$ is subexponential if and only if $F \sim_M G$ (see Embrechts and Goldie [10]).

Our aim is to investigate the financial fragility for two linear portfolios $A = \sum \lambda_i X_i$ and $B = \sum \gamma_i X_i$, where $X_1, X_2, \ldots, X_n$ are independent random variables. From section 3 it is clear that we need to evaluate $P(A > s \cap B > s)$ and $P(A > s \cup B > s)$ for two portfolios A and B as s tends to infinity. The results below show that if the portfolios have assets in common, the tail function of these assets will determine the fragility in case the tail functions are subexponential and of comparable order (i.e. the ratios of the tails are bounded away from zero and infinity).

The case of two portfolios with heavy tailed risk structure and two different assets has been investigated in Geluk and de Vries [18], where the following result was proved.

Theorem 2. *Suppose X, Y are independent random variables with d.f. $F, G \in S$ and $Z = aX + bY$ has d.f.*

$$H_{a,b} \in S \; for \; a, b > 0. \tag{4.19}$$

Suppose $\lambda_1, \lambda_2, \gamma_1, \gamma_2, x, y > 0$ and both coordinates of the point of intersection S of the lines $\lambda_1 x + \lambda_2 y = s$ and $\gamma_1 x + \gamma_2 y = s$ are positive for $s > 0$. Then as $s \to \infty$

(4.20)
$$P(\lambda_1 X + \lambda_2 Y > sx \cup \gamma_1 X + \gamma_2 Y > sy) \sim \overline{F}((\frac{x}{\lambda_1} \wedge \frac{y}{\gamma_1})s) + \overline{G}((\frac{x}{\lambda_2} \wedge \frac{y}{\gamma_2})s).$$

and

(4.21)
$$P(\lambda_1 X + \lambda_2 Y > sx \cap \gamma_1 X + \gamma_2 Y > sy) \sim \overline{F}((\frac{x}{\lambda_1} \vee \frac{y}{\gamma_1})s) + \overline{G}((\frac{x}{\lambda_2} \vee \frac{y}{\gamma_2})s).$$

In case there are assets which are in only one of the portfolios, the asymptotic behavior will be different as is shown in the next result.

Theorem 3. *Suppose X, Y, Z are random variables with distribution functions F, G, H.*

If X, Y, Z are independent and $F, G, H \in S$,

$$F \sim_M H \text{ and } G \sim_M H, \tag{4.22}$$

then

$$P(X + Z > s, Y + \gamma Z > s) = \begin{cases} \overline{H}(s) + o(\overline{F}(s) + \overline{H}(s)) \ if \ \gamma > 1 \\ (1 + o(1))\overline{H}(s) + o(\overline{F}(s) \wedge \overline{G}(s)) \ if \ \gamma = 1 \\ \overline{H}(s/\gamma) + o(\overline{G}(s) + \overline{H}(s/\gamma)) \ if \ 0 < \gamma < 1. \end{cases} \tag{4.23}$$

$$P(X + Z > s, \gamma Z > s) = \begin{cases} \overline{H}(s) + o(\overline{F}(s) + \overline{H}(s)) \ if \ \gamma \geq 1 \\ (1 + o(1))\overline{H}(s/\gamma) \ if \ 0 < \gamma < 1. \end{cases} \tag{4.24}$$

The conditions in Theorems 1 and 2 seem difficult to verify in general. In the corollary below two special cases of importance are listed for which the assumptions are satisfied.

Corollary 1. *Suppose X, Y, Z are independent random variables with distribution functions $F, G, H \in S$ respectively. Suppose at least one of the following is true.*

- *X, Y, Z are identically distributed,*
- *$F, G, H \in L \cap D$.*

Then (4.20),(4.21),(4.23) and (4.24) hold.

If F and G have a regularly varying tail, it follows that $F, G \in L \cap D$, hence the corollary implies that (4.20) and (4.21) hold. In particular, in case X, Y are i.i.d. and in the domain of attraction of an extreme value distribution with $\gamma > 0$, then as $s \to \infty$

$$\begin{aligned} &P(\lambda_1 X + \lambda_2 Y > sx \text{ or } \gamma_1 X + \gamma_2 Y > sy) \\ &\sim \left((\frac{x}{\lambda_1} \wedge \frac{y}{\gamma_1})^{-\alpha} + (\frac{x}{\lambda_2} \wedge \frac{y}{\gamma_2})^{-\alpha} \right) \overline{F}(s) \end{aligned} \tag{4.25}$$

and a similar result for the intersection of the two events.

In order to prove the corollary, we collect some well-known facts in a Lemma. Item (1) is from Embrechts et al. [9], (2) and (3) are from Embrechts and Goldie [10], and item (4) is proved in Geluk [17].

Lemma 2. *Suppose X, Y, Z are independent with distribution functions F, G, H.*

(1) *If $H \in S$ and $\overline{F}(s) = o(\overline{H}(s))$ as $s \to \infty$, then $\overline{F * H}(s) \sim \overline{H}(s)$.*

(2) *If* $F, G \in D \cap S$, *then* $F * G \in D \cap S$.
(3) *If* $F, G \in S$, *then* $F * G \in S$ *if and only if* $F \sim_M G$.
(4) *If* $F, G \in L$ *and* $\sup_x \overline{G}(x)/\overline{F}(x) < \infty$, *then* $F \in S \Leftrightarrow F * G \in S$.

In this case we have $F \sim_M G$, *i.e.*

$$\overline{F * G}(s) \sim \overline{F}(s) + \overline{G}(s) \ (s \to \infty). \tag{4.26}$$

Proof of Corollary 1. Suppose $X, Y < Z$ are i.i.d. random variables with a d.f. in S. It is well known that linear combinations of i.i.d. random variables with a subexponential distribution have a subexponential distribution (see e.g. [18]), i.e. (4.19) holds. Note that (4.22) follows from Lemma 1(3). The second statement follows since (4.19) and (4.22) are satisfied in view of items (2) and (3) in Lemma 1 (note that $D \cap L = D \cap S$). □

Proof of Theorem 2. Suppose X, Y, Z are independent. Write

$$\begin{aligned}
&P(X + Z > s,\ Y + \gamma Z > s) \\
&= \int_{-\infty}^{\infty} \overline{F}(s-x)\overline{G}(s-\gamma x)dH(x) \\
&= \left(\int_{-\infty}^{0} + \int_{0}^{s} + \int_{s}^{\infty}\right) \overline{G}(s-\gamma x)[\overline{F}(s-x) - \overline{F}(s)]dH(x) \\
&+ \overline{F}(s) \int_{-\infty}^{\infty} \overline{G}(s-\gamma x)dH(x) =: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.27}$$

Since $F \in L$, it follows, using dominated convergence, that

$$I_1 = O(\overline{F}(s)\overline{G}(s)) = o(\overline{F}(s)),\ (s \to \infty).$$

Moreover, since $F, H \in L$ and $F \sim_M H$, using Lemma 1 in Geluk and de Vries [18]

$$\begin{aligned}
0 \le I_2 &= \int_0^s \overline{G}(s-\gamma x)[\overline{F}(s-x) - \overline{F}(s)]dH(x) \\
&\le \int_0^s [\overline{F}(s-x) - \overline{F}(s)]dH(x) \\
&= P(X^+ + Z^+ > s) - P(X \vee Z > s) \\
&= P(X + Z > s) - P(X \vee Z > s) + o(\overline{F}(s)) + o(\overline{H}(s)) \\
&= o(\overline{F}(s)) + o(\overline{H}(s)).
\end{aligned} \tag{4.28}$$

where $X^+ = \max(0, X)$.

Note that

$$I_3 = \int_s^\infty \overline{G}(s-\gamma x)\overline{F}(s-x)dH(x) + o(\overline{F}(s)) \\ \leq \overline{H}(s) + o(\overline{F}(s)). \tag{4.29}$$

Note that since $F, G \in S \subset L$, for $\varepsilon > 0$ arbitrary, we may assume w.l.o.g. that $F(0) < \varepsilon$ and $G(0) < \varepsilon$ by adding a constant to X and Y.

In case $\gamma \geq 1$, we obtain a lower estimate as follows

$$\begin{aligned} I_3 &= \int_s^\infty \overline{G}(s-\gamma x)\overline{F}(s-x)dH(x) + o(\overline{F}(s)) \\ &\geq \overline{G}((1-\gamma)s)\int_s^\infty \overline{F}(s-x)dH(x) + o(\overline{F}(s)) \\ &\geq (1-\varepsilon)\overline{F}(0)\overline{H}(s) + o(\overline{F}(s)) + o(\overline{H}(s)) \\ &\geq (1-\varepsilon)^2\overline{H}(s) + o(\overline{F}(s)) + o(\overline{H}(s)). \end{aligned} \tag{4.30}$$

Finally

$$I_4 = \overline{F}(s)P(Y + \gamma Z > s) = o(\overline{F}(s)).$$

Since $\varepsilon > 0$ is arbitrary, the proof for $\gamma > 1$ is complete. The case $\gamma < 1$ can be proved if we replace Z with Z/γ and apply the statement for $\gamma > 1$. If $\gamma = 1$ the role of F and G can be interchanged.

Now suppose $F, H \in S$, $F \sim_M H$ and $\gamma > 1$. Then

$$\begin{aligned} &P(X + Z > s, \gamma Z > s) \\ &= \left\{\int_{-\infty}^0 + \int_0^{s(1-\frac{1}{\gamma})}\right\} P(Z > s - x)dF(x) + \int_{s(1-\frac{1}{\gamma})}^\infty P(Z > \frac{s}{\gamma})dF(x) \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{4.31}$$

Then $I_1 \sim F(0)\overline{H}(s)$ as in the first part of the proof. Moreover $I_2 = \int_0^{s(1-\frac{1}{\gamma})}[(\overline{H}(s-x) - \overline{H}(s)) + \overline{H}(s)]dF(x) = \overline{F}(0)\overline{H}(s) + o(\overline{F}(s)) + o(\overline{H}(s))$, as in the proof of (4.28). Finally $I_3 = \overline{H}(\frac{s}{\gamma})\overline{F}(s(1-\frac{1}{\gamma})) = o(\overline{F}(s)) + o(\overline{H}(s))$ by Lemma 10 in Geluk [17]. The proof in case $\gamma > 1$ is complete. We omit the similar proof of the case $0 < \gamma < 1$. □

We now discuss some examples in which fragility can be calculated explicitly for portfolios with heavy tailed risk structure.

From relation (3.1) it follows that in order to evaluate fragility for two portfolios P and R, it is sufficient to consider the relation

$$E\{\kappa_s|\kappa_s \geq 1\} = \frac{P\{P > s\} + P\{R > s\}}{P\{P > s \text{ } or \text{ } R > s\}}$$

As a consequence, if $\kappa = \lim_{s\to\infty} E\{\kappa_s|\kappa_s \geq 1\}$ exists, then

$$\kappa = 1 + \lim_{s\to\infty} \frac{P\{P \wedge R > s\}}{P\{P \vee R > s\}}. \tag{4.33}$$

We will use the above theorems in order to calculate the fragility index for some examples with two portfolios P and R.

EXAMPLES

(1) Suppose the portfolios have risk structure $P = X + Z$ and $R = Y + Z$, where X, Y and Z are independent with subexponential distributions.

Then, in view of the above Lemma, we may apply Theorem 2 to find that $\kappa = 2$ if $\overline{F}(s) = o(\overline{H}(s))$ and $\overline{G}(s) = o(\overline{H}(s))$, while $\kappa = 4/3$ if $\overline{F}(s) \sim \overline{G}(s) \sim \overline{H}(s)$. It follows in particular that if X, Y, Z are i.i.d. with subexponential distribution, then $\kappa = 4/3$.

(2) Suppose two portfolios have returns $R = \gamma X + (1 - \gamma)Y$ and $Z = (1 - \gamma)X + \gamma Y$, where X, Y are i.i.d. random variables with a regularly varying tail distribution function with exponent $-\alpha$ and $\gamma \in (0, 1)$, $\gamma \neq 1/2$. Then Theorem 1 and (4.33) imply $\kappa = 1 + (\frac{1}{\gamma} - 1)^\alpha$. So the crisis linkage appears to be asymptotically dependent.

(3) Suppose R and Z are defined as in the previous item and $\overline{F}(s) = s^{-\alpha}L_1(s)$, $\overline{G}(s) = s^{-\beta}L_2(s)$ as $s \to \infty$, where L_1, L_2 are slowly varying and $0 < \alpha < \beta$. As before we find $\kappa = 1 + (\frac{1}{\gamma} - 1)^\alpha$.

REMARKS

- Comparing the portfolio in example 3 with that of example 2, one sees that even though for this portfolio the second asset has a return with distribution function which has markedly lighter tails, nevertheless the asymptotic dependency is preserved and at the same level. The return with the heaviest tail drives the fragility. Consider the case of a mixed equity-bond portfolio, which is the standard mix held by most investors. The joint (mis-)fortunes of these portfolios are entirely driven by the equity part. Of course these are only first order results, and one shows that if one looks more precisely into the tail area, the contribution to the joint failure from the bond portfolio will show up in second order terms. Note this also implies that it is not important for the asymptotic dependence results whether the same number of assets are present in both portfolios, which was crucial for the case of discrete returns.

5. ECONOMICS

We now return to the economic issues outlined before and discuss implications of the probability results. We derive copulas which satisfy properties motivated by the economics, we investigate different 4 by 4 banking networks under alternative assumptions on the distribution of the asset returns and calculate the FI index and rank order their potential for systemic risk, we briefly deal with non-linear instruments, discuss sunspot equilibria and close by investigating the effect of time dependency for the results.

5.1. **Copulas.** Copulas have gained in popularity as a measure of dependency in economics and finance due to the dismay over the standard use of correlation. In case of a bivariate normal distribution, the correlation measure aptly summarizes the dependency derived from the Gaussian copula. But over the years a lot of evidence has been accumulated showing that the normal fits badly to financial return series, the question is whether such a misfit also applies to the dependency structure. Hence researchers in finance have started to use other copulas. For this reason it appears important to connect the concept of a copula to our measure. Recall that our measure only has something to say about the tail region, while a copula is a global dependency measure. By taking limits, the two concepts can be connected

$$\lim_{s\to\infty} E\left\{\kappa|\kappa\geq 1,s\right\} = \lim_{p\uparrow 1}\frac{2(1-p)}{1-C(p,p)}. \tag{5.1}$$

Coming back to the examples of parametric copulas, we have the following results. We give results for the right hand side limit and leave the left hand side to the reader. Starting with the independent copula $C(p,p)=p^2$,it is immediate that

$$\lim_{p\uparrow 1}\frac{2(1-p)}{1-C(p,p)} = \lim_{p\uparrow 1}\frac{2(1-p)}{(1+p)\,(1-p)} = 1.$$

For the Morgenstern copula, while the correlation is non-zero when $\delta \neq 0$, nevertheless

$$\begin{aligned}
\lim_{p\uparrow 1} \frac{2(1-p)}{1-C(p,p)} &= \lim_{p\uparrow 1} \frac{2(1-p)}{1-p^2[1+\delta(1-p)^2]} \\
&= \lim_{p\uparrow 1} \frac{2(1-p)}{1-p^2} \\
&= \lim_{p\uparrow 1} \frac{2}{1+p} \\
&= 1
\end{aligned}$$

Similarly, one can verify that the bivariate normal distribution gives asymptotic independence (see previous results).

For the Logistic copula, we have

$$\begin{aligned}
\lim_{p\uparrow 1} \frac{2(1-p)}{1-C(p,p)} &= \lim_{p\uparrow 1} \frac{2(1-p)}{1-\exp\{-[2(\ln \frac{1}{p})^{1/\beta}]^{\beta}\}} \\
&= \lim_{p\uparrow 1} \frac{2(1-p)}{1-p^{2^{\beta}}} \\
&= 2^{1-\beta}
\end{aligned}$$

which is larger than one for $\beta < 1$. Similarly, for the Pareto copula

$$\begin{aligned}
\lim_{p\uparrow 1} \frac{2(1-p)}{1-C(p,p)} &= \lim_{p\uparrow 1} \frac{2(1-p)}{2-2p-[2(\frac{1}{1-p})^{1/\alpha}-1]^{-\alpha}} \\
&= 1+\lim_{p\uparrow 1} \frac{(1-p)[2-(1-p)^{1/\alpha}]^{-\alpha}}{2(1-p)-(1-p)[2-(1-p)^{1/\alpha}]^{-\alpha}} \\
&= 1+\lim_{p\uparrow 1} \frac{(1-p)[2]^{-\alpha}}{2(1-p)-(1-p)[2]^{-\alpha}} \\
&= 1+\frac{1}{2^{1+\alpha}-1}
\end{aligned}$$

Thus these two copulas induce asymptotic dependence.

Lastly, for the case of the Plackett copula we find

$$\lim_{p\uparrow 1} \frac{2(1-p)}{1-C(p,p)}$$

$$
\begin{aligned}
&= \lim_{p\uparrow 1} \frac{2(1-p)}{1-\frac{1}{2(\theta-1)}[1+2(\theta-1)p-\sqrt{[1+(\theta-1)2p]^2-4\theta(\theta-1)p^2}]} \\
&= \lim_{p\uparrow 1} \frac{2(1-p)}{(1-p)-\frac{1}{2(\theta-1)}[1-\sqrt{1+4(\theta-1)p-4(\theta-1)p^2}} \\
&= \lim_{p\uparrow 1} \frac{2(1-p)}{(1-p)-\frac{1}{2(\theta-1)}[1-\sqrt{1+4(\theta-1)p(1-p)}} \\
&= 2.
\end{aligned}
$$

Thus this copula induces perfect dependence in the tail areas.

Let us now come back to the example portfolio and derive an economically motivated property of the copula. Recall that in the simple example, the banks' portfolios are

$$Q=(1-\gamma)X+\gamma Y, \tag{5.2}$$

and

$$W=\gamma X+(1-\gamma)Y. \tag{5.3}$$

Let X and Y be i.i.d. random variables with regularly varying symmetric tails, i.e. as $s\to\infty$

$$P\{X\le -s\}=P\{Y\le -s\}=P\{X>s\}=P\{Y>s\}=s^{-\alpha}L(s).$$

For the marginal distributions adapt Feller's (1971, VIII.8) convolution theorem, to show that

$$
\begin{aligned}
P\{(1-\gamma)X+\gamma Y &> s\}=P\{\gamma X+(1-\gamma)Y>s\} \\
&= [\gamma^\alpha+(1-\gamma)^\alpha]s^{-\alpha}L(s).
\end{aligned}
$$

Suppose $\gamma\in(1/2,1)$. For the joint distribution function $P\{Q\le s,W\le t\}$,one has (by the above) as $s,t\to\infty$

(5.4)

$$P\{Q\le s,W\le t\}=\begin{cases} 1-[\gamma^\alpha+(1-\gamma)^\alpha]t^{-\alpha}L(t) & \text{as } \frac{s}{t}>\frac{\gamma}{1-\gamma} \\ 1-\gamma^\alpha[s^{-\alpha}L(s)+t^{-\alpha}L(t)] & \text{as } \frac{1-\gamma}{\gamma}\le\frac{s}{t}\le\frac{\gamma}{1-\gamma} \\ 1-[\gamma^\alpha+(1-\gamma)^\alpha]s^{-\alpha}L(s) & \text{as } \frac{s}{t}<\frac{1-\gamma}{\gamma} \end{cases}.$$

Note that the two portfolio lines of the portfolios Q and W, cut the axes at $(s/(1-\gamma),0)$ and $(t/\gamma,0)$ along the x-axis, while along the y-axis these points are respectively $(0,s/\gamma)$ and $(0,t/(1-\gamma))$. The intuition behind the probabilities in (5.4) is as follows. To a first order, it is only the mass along the axes above the points where the portfolio lines cross the axes closest to the origin which contributes to the probability. It is then immediate that

$$\begin{aligned}\lim_{s\to\infty} E\{\kappa|\kappa\geq 1\} &= \\ \lim_{s\to\infty}\frac{2[\gamma^\alpha+(1-\gamma)^\alpha]s^{-\alpha}L(s)}{2\gamma^\alpha s^{-\alpha}L(s)} &= 1+(\frac{1}{\gamma}-1)^\alpha.\end{aligned}$$

Thus the two portfolios returns Q and W are asymptotically dependent.

Alternatively, we verify the limit on the right hand side in (5.1). The copula associated with $P\{Q\leq s, W\leq t\}$ should satisfy

(5.5)

$$C_{Q,W}(x,y)=\begin{cases} y & \text{as } \frac{1-y}{1-x}>(\frac{\gamma}{1-\gamma})^\alpha \\ 1-\frac{\gamma^\alpha}{\gamma^\alpha+(1-\gamma)^\alpha}[2-x-y] & \text{as } (\frac{1-\gamma}{\gamma})^\alpha\leq\frac{1-y}{1-x}\leq(\frac{\gamma}{1-\gamma})^\alpha \\ x & \text{as } \frac{1-y}{1-x}<(\frac{1-\gamma}{\gamma})^\alpha \end{cases}$$

for x and y in a neighborhood of 1. Note that if α is close to zero, $C_{Q,W}(x,y)$ is close to the perfect dependent copula $\min(x,y)$, while when α is very large $C_{Q,W}(x,y)$ is close to $x+y-1$, which for x,y in a neighborhood of 1 is the first order Taylor approximation of the independent copula. It readily follows that

$$\begin{aligned}\lim_{p\uparrow 1}\frac{2(1-p)}{1-C(p,p)} &= \lim_{p\uparrow 1}\frac{2(1-p)}{1-[1-\frac{\gamma^\alpha}{\gamma^\alpha+(1-\gamma)^\alpha}2(1-p)]} \\ &= 1+(\frac{1}{\gamma}-1)^\alpha.\end{aligned}$$

The upshot of all this is that economically relevant copulas may be quite different form the popular functional forms from the literature. Note how the (5.5) differs from the standard type of copulas. The use of the latter is mainly governed by the ease to which these lend themselves to econometric estimation. But this may not always yield the economically relevant specification. For example, if one uses the Gaussian copula or the Morgenstern copula, one automatically rules out that the random variables can be asymptotically dependent, while the Plackett copula induces to much dependence. For the economic problem of systemic risk in the financial sector, we now at least have a theory about the relevant economic functional form.

5.2. **options.** Consider how the analysis has to adapted when the portfolio includes non-linear instruments like an option. Note this does not necessarily destroy the linearity of a portfolio. Suppose a call option is included, but not the underlying asset. Then, even though the call is non-linearly related to the underlying security, for such an portfolio it is only the return to the option which counts, and this enters linearly. Also note that standard option theory holds that over a short span of time the option and the underlying are linearly related to the market factor, where in case of the option the option's elasticity enters next to the beta of the underlying. This still permits an analysis within the above framework.

Suppose, however, that in one portfolio the option is held till maturity and that another portfolio comprises the underlying, then one has to take care of the non-linearities due to the fact that the option may be out of the money. Let S_t denote the share price at time t and let C_t denote the price of the (European) call option. Suppose the call is at the money at the time of purchase t. The option expires at time $T > t$. If held till maturity, the gross return on the underlying and the call are respectively

$$\frac{S_T}{S_t} \text{ and } \max[0, (\frac{S_T}{S_t} - 1)\frac{S_t}{C_t}].$$

Suppose that S_T/S_t follows a (continuous) distribution $F(S_T/S_t)$ say, for which the left tail is in the domain of attraction of the Weibull distribution, and

$$\lim_{s \downarrow 0} P\{\frac{S_T}{S_t} \leq s\} = 0,$$

while $F(1) > 0$ so there is a non-zero probability that the option ends out of the money. The option at expiration yields the return $\max[0, (\frac{S_T}{S_t} - 1)\frac{S_t}{C_t}]$ and where $S_t/C_t > 1$. It follows that

$$\lim_{s \downarrow 0} P\{\max[0, (\frac{S_T}{S_t} - 1)\frac{S_t}{C_t}] \leq s\} = F(1).$$

Note moreover that the joint probability for $s < 1$ collapses to

$$\lim_{s \downarrow 0} P\{\frac{S_T}{S_t} \leq s, \max[0, (\frac{S_T}{S_t} - 1)\frac{S_t}{C_t}] \leq s\} = P\{\frac{S_T}{S_t} \leq s\}.$$

Since if $S_T < S_t$, the option return is zero and the hence $\max[0, (\frac{S_T}{S_t} - 1)\frac{S_t}{C_t}] \leq s < 1$ is automatically satisfied. It follows that $FI = 1$, ie the two portfolios are asymptotically independent.

To give one other example, consider two portfolios each consisting of an at the money call option, where each option is written on a different stock. Suppose the two stock returns are independently distributed, with distribution functions $F(.)$ and $G(.)$. Thus the loss returns on the two stocks are evidently asymptotically independent. Assume again that the two stock returns have continuous distributions and that the there is a non-zero probability that the stock prices fall below their level at the date of purchase of the option, ie $F(1) > 0$, and $G(1) > 0$.[9]

[9]The options have the same expiration date and are bought on the same date.

It follows that for the two option portfolios

$$\lim_{s\downarrow 0} E\left\{\kappa | \kappa \geq 1, s\right\} = \frac{1}{1 - \frac{F(1)G(1)}{F(1)+G(1)}} > 1.$$

Since $F(1)G(1) > 0$. Thus the option portfolios are asymptotically dependent even though the stock portfolios are asymptotically independent!

5.3. **sequence of networks and large portfolios.** In this section we investigate how different dependence measures and different tail shapes affect the ranking in of systems regarding systemic risk. The network configurations we discuss are motivated by the cases discussed in the theoretical economic literature on systemic risk, such as Allen and Gale (2000) and Freixas, Parigi and Rochet (2000).

Suppose there are four projects: $4U$, $4X$, $4Y$, and $4T$. The finance of each project can be broken down into four equally sized participations. There are also four distinct banks: B_1, B_2, B_3, and B_4. Consider the following cases of sysndicated loans.

Case 1. *Each bank finances one entire project. In particular*

$$B_1 = 4U,\ B_2 = 4X,\ B_3 = 4Y,\ B_4 = 4T,$$

where we identify the portfolio of each bank with its name tag B_i.

Case 2. *Each bank participates in two projects. The specific portfolios are*

$$B_1 = 2U + 2X, B_2 = 2X + 2Y,\ B_3 = 2Y + 2T,\ B_4 = 2T + 2U.$$

Case 3. *There is further diversification. In particular, the portfolios are*

$$B_1 = 2U+X+Y, B_2 = 2X+Y+T,\ B_3 = 2Y+T+U,\ B_4 = 2T+U+X.$$

Case 4. *All bank portfolios are fully diversified:*

$$B_i = U + X + Y + T,\ \text{for } i = 1, ..., 4.$$

Note that this represents different network configurations as discussed in Allen and Gale (2000) and Freixas, Parigi and Rochet (2000), but now for loan syndication rather than the interbank market. In particular the last portfolio is remniscent to the diversified lending case and the second portfolio resembles the credit chain funding of Freixas et al. (2000).

5.3.1. *normal.* Suppose the project returns U, X, Y, and T are standard normally distributed. In that case the correlation matrix C is a natural representation of the network dependencies. Since the multivariate normal distribution implies asymptotic independence, we seek a measure other than $E\{\kappa|\kappa \geq 1\}$ which can distinguish between the different network dependencies. As a measure for the dependencies we propose to use $\mathbb{D} = \text{trace}\, CC^T$. In case one the $C = I$ the identity matrix and hence $\mathbb{D}$ =4.

In the second case

$$C = \begin{pmatrix} 1 & 1/2 & 0 & 1/2 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 1/2 & 0 & 1/2 & 1 \end{pmatrix},$$

so that $\mathbb{D}$ =6. Note that banks are only partially connected, which is reflected through the zero's in the correlation matrix.

For case three the zeros in the previous matrix are replaced by 2/3.

$$C = \begin{pmatrix} 1 & 1/2 & 2/3 & 1/2 \\ 1/2 & 1 & 1/2 & 2/3 \\ 2/3 & 1/2 & 1 & 1/2 \\ 1/2 & 2/3 & 1/2 & 1 \end{pmatrix}.$$

This implies that $\mathbb{D} = 7\frac{7}{9}$.

The fully diversified portfolios imply that all banks become perfectly correlated, so that $\mathbb{D}$ =16. Note that the measure nicely reflects the increases in network connectedness as we move from first case to the last case.

5.3.2. *fat tails.* Now suppose the project returns are all heavy tail distributed. In particular, suppose

$$\Pr\{U > s\} = \Pr\{X > s\} = \Pr\{Y > s\} = \Pr\{T > s\} = as^{-\alpha},\ \alpha > 0,\ s > 0.$$

Hence $E\{\kappa|\kappa \geq 1, s\}$ is a natural measure for the extreme network dependencies. It is immediate that for case 1

$$\lim_{s\to\infty} E\{\kappa|\kappa \geq 1\} = 1.$$

Note that FI= 1 as well.

For the case 2 use Feller's convolution theorem as s becomes large

$$\Pr\{2U + 2X > s\} = 2a2^{\alpha}s^{-\alpha}.$$

Moreover by the arguments above, the probability of no failures

$$\Pr\{2U + 2X \leq s, 2X + 2Y \leq s, 2Y + 2T \leq s, 2T + 2U \leq s, \}$$

can be found by noting that

$$(2U + 2X = s, 2X + 2Y = s, 2Y + 2T = s, 2T + 2U = s)$$

spans a hyperplane which cuts the four axes at the points

$$(\frac{s}{2}, 0, 0, 0); (0, \frac{s}{2}, 0, 0); (0, 0, \frac{s}{2}, 0); (0, 0, 0, \frac{s}{2}).$$

Above each of the these points there is mass $a2^{\alpha}s^{-\alpha}$ along the axes. The mass above the four points together gives the probability

$$\begin{aligned} 1 - \Pr\{2U + 2X &\leq s, 2X + 2Y \leq s, 2Y + 2T \leq s, 2T + 2U \leq s, \} \\ &= 4 \times a2^{\alpha}s^{-\alpha}. \end{aligned}$$

Combining gives

$$\lim_{s\to\infty} E\{\kappa|\kappa \geq 1\} = \frac{4 \times 2a2^{\alpha}s^{-\alpha}}{4 \times a2^{\alpha}s^{-\alpha}} = 2.$$

Thus the index $FI = 1/3$ in this case.

The third case is interesting since the denominator is as in the second case. Note that

$$(2U + X + Y = s, 2X + Y + T = s, 2Y + T + U = s, 2T + U + X = s)$$

spans the same (inner) hyperplane through the points

$$(\frac{s}{2}, 0, 0, 0); (0, \frac{s}{2}, 0, 0); (0, 0, \frac{s}{2}, 0); (0, 0, 0, \frac{s}{2})$$

as in the previous case. Since we need the probability to be below triangular planes like

$$2U + X + Y = s,$$

the binding point is which is closest to the origin, i.e. along the axis where the triangle cuts at $s/2$. Thus

$$\begin{aligned} 1 - \Pr\{2U + X + Y &\leq s, 2X + Y + T \leq s, 2Y + T + U \leq s, 2T + U + X \leq s, \} \\ &= 4 \times a2^{\alpha}s^{-\alpha}. \end{aligned}$$

The numerator is straightforward by Feller's convolution result and equals

$$4 \times \Pr\{2U + X + Y > s\} = 4(2^{\alpha} + 2)as^{-\alpha}.$$

Hence, the third network implies

$$\lim_{s\to\infty} E\{\kappa|\kappa \geq 1\} = \frac{4(2^{\alpha} + 2)as^{-\alpha}}{4a2^{\alpha}s^{-\alpha}} = 1 + \frac{1}{2^{\alpha-1}}.$$

From this it follows that $FI = \frac{2}{3}\frac{1}{2^{\alpha}}$. Note that for $\alpha > 1$, this network is less fragile than the previous one.

Lastly, with full diversification, the four portfolios become totally dependent so that

$$\lim_{s\to\infty} E\left\{\kappa|\kappa \geq 1\right\} = 2$$

and $FI = 1$.

We compare the sequence of networks by their ranking of systemic dependencies. If we use the correlation matrices, we see that the networks become increasingly more interdependent and exposed to systemic risk if we use the $\mathbb{D}$ measure. A different picture emerges once we use the concept of asymptotic dependence. Indeed, for networks 1, 2 and four, the measure

$$E\left\{\kappa|\kappa \geq 1, s\right\}$$

is increasing. But the third network has a lower asymptotic dependency measure than the second network as long as the first moment is bounded, i.e. $\alpha > 1$.

5.4. **sunspots.** The theoretical economics literature also devotes a lot of attention to the issue of multiple equilibria and how agents coordinate on these equilibria. The starting point of the literature is Keynes famous beauty contest explaining stock prices. With multiple equilibria fundamentals do not fully determine outcomes, somehow one of the equilibria is being played by coordination on a sunspot. We now show that this approach can also be subsumed under our reduced form approach. Suppose that the fundamentals induce multiple equilibria. Take for example the Diamond and Dybvig (1983) model of liquidity preference model of banking. In this model there are two Nash equilibria, one with and the other without a bank run. On a macro level coordination is sometimes obtained through a promise by the central bank to bail out illiquid but solvent banks. But this is widely recognized to trigger moral hazard on part of the bank by engaging in riskier projects, so that the promise is usually not made so ostensibly (some advocate a mixed strategy). Extensions of the Nash theory sometimes provide more or less reasonable equilibrium selection devices as in the case of the global game equilibrium concept. The theory usually provides no explicit mechanism about how agents coordinate and switch between sunspot equilibria. Mostly it is assumed that agents coordinate via the exogenous device. This sunspot is a random event observed by all agents, which signals to them it is time to change and play another equilibrium. The sunspot is a random variable that has no direct effect on the economy (ie is unrelated to the fundamentals).

In our reduced form approach, we do not concern ourselves with the precise game being played, but directly focus on the set of fundamentals. To incorporate the sunspot equilibria we can just proceed as before and write down a linear set of (differently weighted, or including different) fundamentals that support the two equilibria. The switch between the equilibria is exogenous to the theory. To model this we assume there is a Bernoulli random variable representing the sunspot that indicates which of the two equilibria is relevant. This may appear quite crude, more intricate models like a Markov switching model etc. can be imagined, but these do not essentially alter the results.

In particular, consider the sunspot model. Consider three independently distributed asset returns X, Y and Z. Let the portfolio returns be $Q = X + Y$ with probability π, and $Q = Y$ with probability $1 - \pi$. Similarly, let $W = X + Z$ with probability π, and $W = Z$ with probability $1 - \pi$. Note that with probability $1 - \pi$ the portfolio returns are independent. All three assets X, Y and Z are iid dsitributed with Pareto type tails

$$P\{X > s\} = P\{Y > s\} = P\{Z > s\} = cs^{-\alpha}.$$

Using Feller's theorem one calculates that

$$\begin{aligned} P\{Q &> s\} = \pi P\{X + Y > s\} + (1 - \pi)P\{Y > s\} \\ &= \pi 2cs^{-\alpha} + (1 - \pi)cs^{-\alpha} \\ &= (1 + \pi)cs^{-\alpha} \end{aligned}$$

For the joint probability, using the theoretical results gives

$$\begin{aligned} P\{Q &\leq s, W \leq s\} = \pi P\{X + Y \leq s, X + Z \leq s\} + (1 - \pi)P\{Y \leq s, Z \leq s\} \\ &\approx \pi[1 - 3cs^{-\alpha}] + (1 - \pi)[1 - 2cs^{-\alpha}] \\ &= 1 - (\pi + 2)cs^{-\alpha}. \end{aligned}$$

where in the second step we omit terms which are second order small. We can compute our measure of dependence

$$\begin{aligned} E\left\{\kappa | \kappa \geq 1\right\} &= \lim_{s\to\infty} E\left\{\kappa | \kappa \geq 1, s\right\} \\ &= \lim_{s\to\infty} \frac{P\{Q > s\} + P\{W > s\}}{1 - P\{Q \leq s, W \leq s\}} \\ &= \lim_{s\to\infty} \frac{2(1 + \pi)cs^{-\alpha}}{(\pi + 2)cs^{-\alpha}} \\ &= 1 + \frac{\pi}{2 + \pi}. \end{aligned}$$

The $FI = \pi/(4 + 2\pi)$.

5.4.1. *time dependency, arch.* In case the factors or returns in our linear portfolios are time dependent, this does not affect the results in the paper. The crucial point is that all our results depend on the cross sectional dependency induced by the portfolios, but do not depend on the structure of the stochastic processes. For example, consider the two portfolios $Q_t = (1-\gamma)X_t + \gamma Y_t$ and $W_t = \gamma X_t + (1-\gamma)Y_t$. Again X_t and Y_t are independent, but each follows some stochastic process. Specifically, suppose that X_t and Y_t follow two independent ARCH(1) processes, with the same parameter values. Since de Haan (1989) showed that the stationary distribution of an ARCH(1) process is in the domain of attraction of a Frechet limit law, it follows again that $\{Q_t, W_t\}$ are asymptotically dependent.

6. Conclusion

Banking systems are well known to be inherently unstable. Thus there is need for a measure of the potential for systemic breakdown. It is well understood that the correlation measure may not acurately reflect the risk of joint breakdowns. To remedy this deficiency we have constructed a measure FI which accurately measures the severity of joint tail risk. The FI measure reveals the type of tail dependence (asymptotic dependence or independence), which either implies strong or weak fragility of the system, and also says how much there is of such dependence. Banks' portfolios are essentially linear in their exposures, either directly through the portfolio asset and liability returns or indirectly through the relation with macro risk factors. This permits an evaluation of the FI under a very wide range of asset return distributions. Assuming that the marginal distributions of the risk drivers are in the domain of attraction of an (univariate) extreme value limit law, we could show when the system would be weakly and when it would be strongly fragile. For example, discrete returns and with all types of assets present in all bank portfolios imply strong fragility. Under continuous discounting, normal distributed returns induce weak fragility, but Student-t type returns render the system strongly fragile. Subsequently we studied different kind of banking networks and their fragility in terms of FI and used the arguments to construct the characteristics of economically relevant copula. It is hoped that this characterization of the financial fragility will help in bridging the gap between theory and practice and be especially helpful in evaluating the macro prudential aspects of bank supervision.

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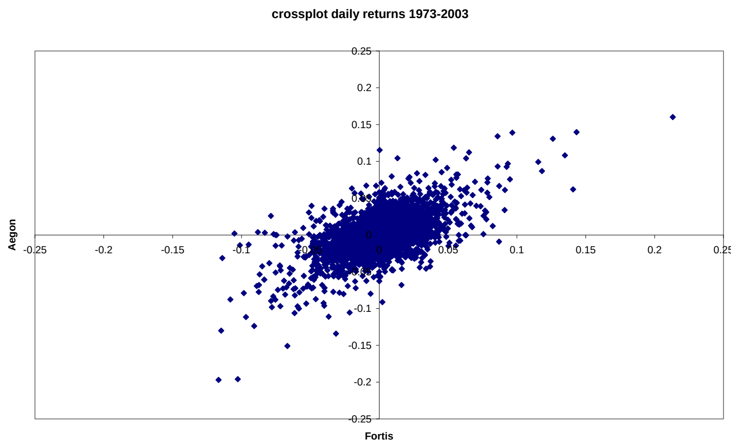
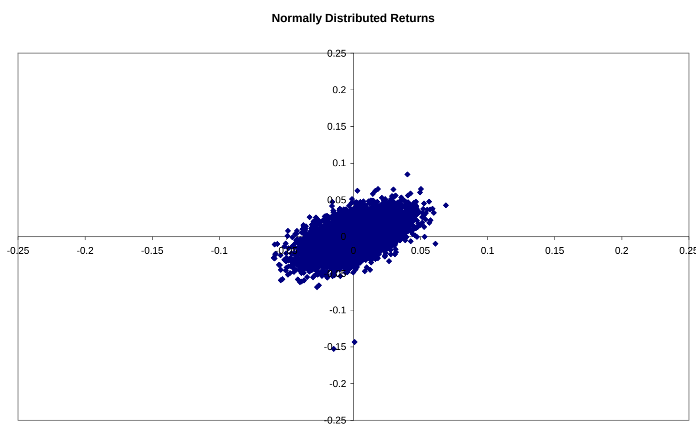
7. Appendix

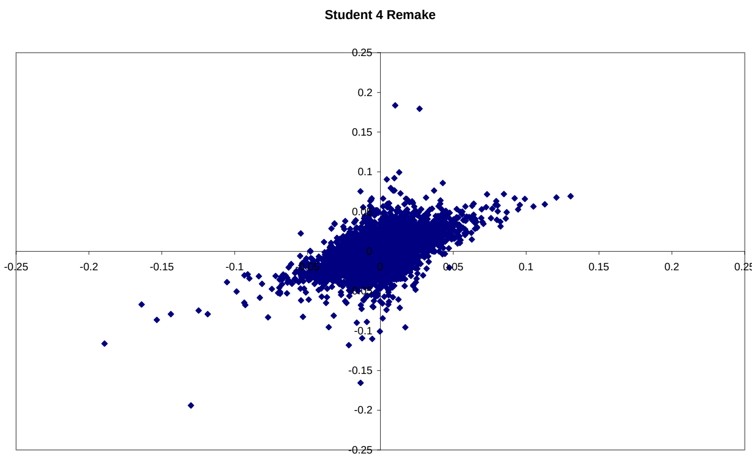
To motivate the issue of extreme dependencies, consider the following crossplot for the daily log stock returns of two Belgium-Dutch financial conglomerates from 1973 to 2003. The two stocks are clearly not independent, the correlation measure is $\rho = 0.57$. To be able to interpret this correlation we determine whether the standard finance bivariate normal model is applicable. We used the estimated correlation coefficient ρ together with the means and variances of the two marginal distributions and generate an equal amount of bivariate normally distributed pseudo random numbers. The sample is shown in the next figure giving an almost perfect ellipse except for two outliers along the negative y-axis. The plots differ markedly in the extreme North-East and South-West corners. The true data contain many more outliers, which moreover are located along the diagonal, and thus occur jointly.





Interestingly, the normal resample contains some outliers, but these are locate along the y-axis, instead of lying close to the diagonal as in the true data. Thus we cannot use the normal model to interpret the correlation coefficient and dependency in the tail areas.

The observed discrepancy between the two figures is not atypical for asset markets in general. The empirical literature has concluded the concept of normal based correlation does not adequately capture the dependency structure for the financial sector. Suppose instead that the data come from a bivariate Student-t distribution with 4 degrees of freedom as simulated in the next figure. The Student model better captures the outliers which we find in the real data, although the disk of the Student model is a bit is more like a diamond than an ellipse in the center part.



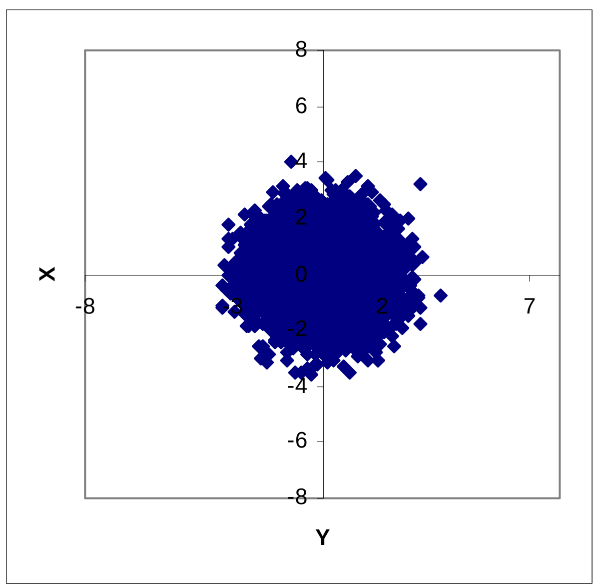


FIGURE 1. Independent Normally distributed random variables

To give a simple explanation for the observed differences, consider the two asset returns X and Y which are either independently normally distributed, or Student-t distributed, as depicted in the next two figures.

Note that the normal data form a nice round disk, but that the Student data are more plus-shaped.

Next we form two portfolios from these data: $X - Y$ and $X + Y$. Note that the two portfolios are uncorrelated, regardless of whether

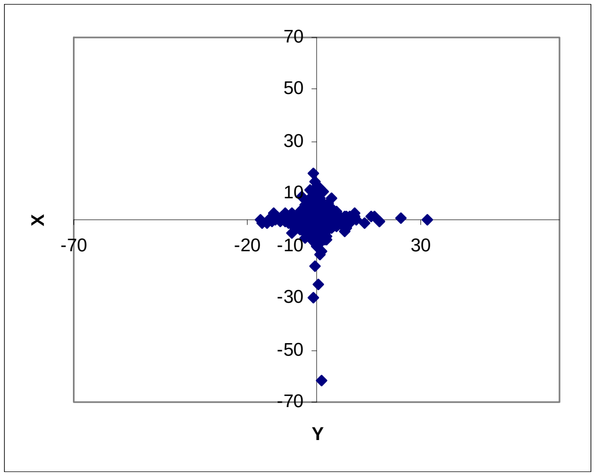


FIGURE 2. Independent Student distributed random variables

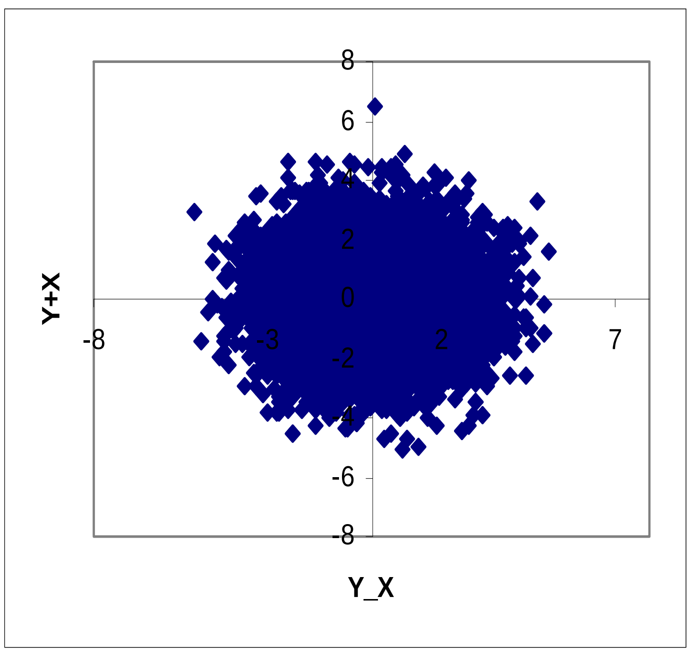


FIGURE 3. Zero correalted Normally distributed random variables

X and Y are normally distributed or Student-t. The next plot gives the portfolio plot for the normal data. Since X and Y are normally distributed, the portfolios are independent and we obtain again a round disk. The last plot gives the portfolio plot for the Student data, which

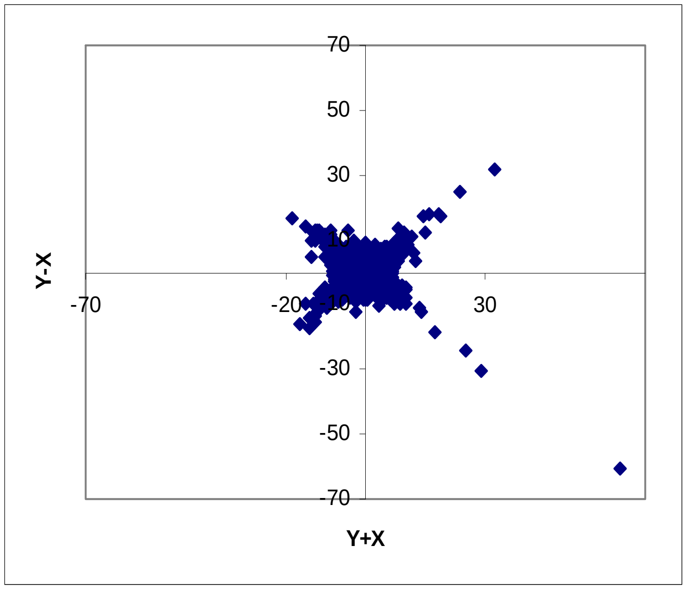


FIGURE 4. Zero correlated Student distributed Random Variables

is x-shaped. Even though uncorrelated, one easily sees the portfolios are not independent. In fact what this picture achieves is to rotate the plus-shaped outliers from the first plot and deposit these along the diagonals, preserving zero correlation, but inducing a lot of dependence in the four tail regions.

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